Lecture 6 : Radiation, Energy Transport and Opacity

6.1 THE RADIATION FIELD

Consider a small area element, dA, from which radiation of frequency ν is emerging into a solid angle $d\Omega$ along in a direction at angle θ relative to the normal vector $\hat{n}$ to the surface, in a time dt. We can describe this radiation by a **monochromatic intensity**, I_ν which is related to the energy E_ν of the radiation by

$$E_\nu = I_\nu d\Omega dt \cos\theta d\nu dA \tag{6.1}$$

where the units of the intensity is erg cm^{-2} sec^{-1} sr^{-1} Hz^{-1}. The **energy flux**, F_ν is defined by

$$dE_\nu = F_\nu dA dt \tag{6.2}$$

and is the quantity of radiation energy passing through the area dA in time dt. It is easy to show by constructing concentric spheres and considering the amount of energy flowing through their surfaces that flux falls of with distance in proportion to the square of the distance, i.e. it follows the inverse square law.

The energy density of the radiation associated with this infinitesimal beam is given by the energy of the beam per unit of distance traveled. Since the radiation travels at the velocity of light, c, the distance dl traveled in time dt is $dl = cdt$, so that

$$E_\nu = \frac{I_\nu}{c} d\Omega dt d\nu dV \tag{6.3}$$

where dV is the small volume element defined by $\cos\theta dA dl$.

For a sufficiently small volume element, dV, I_ν is sufficiently close to a constant, and so the energy density of the field u_ν is given by the integral of the intensity over solid angle

$$u_\nu = \frac{1}{c}\int I_\nu(\Omega) d\Omega. \tag{6.4}$$

The radiation carries momentum, and therefore pressure. The momentum p of a photon is given by $p = E_\nu/c = h\nu/c$, where h is the Planck constant.

The momentum flux (i.e. the pressure) of I_ν is therefore I_ν/c, and the component of this along the normal to dA is $I_\nu \cos\theta/c$. Further, the flux due to the effective area of dA is $dA\cos\theta$, so that the pressure of the radiation field normal to dA is

$$dP_\nu = \frac{I_\nu}{c}\cos^2\theta. \tag{6.5}$$

The total pressure along the normal is thus the integral over solid angle,

$$P_\nu = \int \frac{I_\nu}{c}\cos^2\theta d\Omega. \tag{6.6}$$

A useful quantity is the **mean intensity** J_ν of the field, which is the integral of the intensity I_ν over solid angles (i.e. the average I_ν)

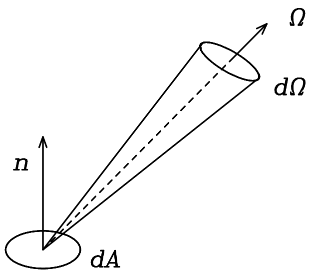

Figure 6.1. Radiation emergent from a small area element dA with normal vector $\hat{n}$, passing into a small solid angle $d\Omega$.

$$J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega. \tag{6.7}$$

In an **isotropic** radiation field, the flux is the same in all directions. In this case $J_\nu = I_\nu$. For such a field I_ν is independent of Ω.

Problem 6.1 Show that the energy density u_ν of an isotropic field is

$$u_\nu = \frac{4\pi I_\nu}{c} \tag{6.8}$$

and hence that the pressure of the radiation field is related to the energy density by

$$P_\nu = \frac{u_\nu}{3}. \tag{6.9}$$

This result can then be integrated over frequency ν so that we get

$$P = \frac{u}{3}. \tag{6.10}$$

6.2 ENERGY ABSORPTION AND EMISSION

In a physical medium, an infinitesimal beam of light, represented by I_ν travels along a direction l, and will lose energy due to absorption processes (i.e. via its interaction with the matter) and gain energy from emission processes (due to energy production by the matter or by scattering oflight into the beam). Absorption is described by the medium *opacity.*

6.2.1 Opacity

Energy may be lost along the beam due to being absorbed by the matter itself, or it may be scattered out of the beam. We can characterise these processes by the **opacity** of the medium, k_ν, which in general will be a function of the wavelength, ν. This quantity is defined in terms of the **mean**

free path of a photon l_ν, or the average distance a photon travels along the beam between interactions with the matter.

$$l_\nu = \frac{1}{k_\nu}. \tag{6.11}$$

Alternatively, the opacity of the material can also be expressed as a cross-section, σ_ν, which is defined by

$$k_\nu = n\sigma_\nu \tag{6.12}$$

where n is the number density of absorbing particles.

Problem 6.2 In the center of the Sun, the opacity of the material is due primarily to *electron scattering.* Assuming a density $\rho = 100$ g cm^{-3} and cross-section $\sigma = 0.6 \times 10^{-24}$ cm^2, what is the mean free path l of the photons?

We now define a dimensionless quantity, τ_ν, which is called the **optical depth** of the matter. This again will in general depend on the frequency, since matter absorbs radiation with different efficiency at different wavelengths. The optical depth is given by the ratio of the distance traveled ds along the beam relative to the mean free path, and is defined differentially by

$$d\tau_\nu = k_\nu ds = \frac{ds}{l_\nu}. \tag{6.13}$$

The optical depth along a particular direction s tells us about the *transparency* of the medium, since when $\tau_\nu = 1$, the photons are likely to undergo interactions with the medium and do not travel freely within it.

If $\tau \gg 1$, a medium is termed **optically thick**, which means that photons will undergo many scatterings within the material. If $\tau \ll 1$, the medium is termed **optically thin**. Photons traveling through an optically thin medium do so freely and undergo very few interactions with it.

The amount of intensity I_ν removed from the beam along a distance ds is thus

$$dI_\nu = -I_\nu k_\nu ds = -I_\nu d\tau_\nu. \tag{6.14}$$

Problem 6.3 Integrate this equation to show that the intensity of a beam is reduced by a factor e when the optical depth $\tau = 1$.

6.2.2 Emission

The matter may also emit photons, adding them to the beam traveling through it. We denote this emission by j_ν, this being the amount of energy emitted per unit time, per unit frequency, per unit solid angle into some direction, Ω. Hence the amount of intensity added to a beam dI_ν is

$$dI_\nu = j_\nu ds. \tag{6.15}$$

6.3 EQUATION OF RADIATIVE TRANSFER

The change in the intensity dI_ν of a beam of light along a length ds due to absorption and emission follows from the above and is given by

$$\frac{dI_\nu}{ds} = -k_\nu I_\nu + j_\nu. \tag{6.16}$$

This is called the **equation of radiative transfer**. In general, it will depend on the direction Ω under consideration. The two terms here represent, firstly, the proportional amount removed from the beam, and secondly, the amount of energy emitted into the beam (either via energy generation processes in the material, or by scattering of photons).

An alternative form is obtained by using the definition of the optical depth, $d\tau_\nu = k_\nu ds$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{j_\nu}{k_\nu} = -I_\nu + S_\nu \tag{6.17}$$

where $S_\nu = j_\nu/k_\nu$ is termed the **source function**.

This equation describes the radiation field along a beam, which in general is at some angle, θ, to the normal to the surface element dA (see figure 6.1). Hence, along some direction of interest, we have the projection onto the normal

$$\mu\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{6.18}$$

where $\mu = \cos\theta$.

Multiplying this equation by μ and integrating over solid angle $d\Omega = \sin\theta d\theta d\phi$, one can show that the flux F_ν obeys

$$\pi F_\nu = -c\frac{dP_\nu}{d\tau_\nu}. \tag{6.19}$$

Thus the radiation flux is proportional to the gradient of the radiation pressure.

6.4 BLACK-BODY RADIATION

Photons emitted by a body which is at a constant temperature, T have a particular spectrum (i.e. intensity I as a function of frequency ν), which is called the **black-body spectrum**. The spectral form was calculated by Planck in the 19th century, and is a consequence of assuming that light has a quantum nature. The Planck function is

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1} \tag{6.20}$$

where $h = 6.626 \times 10^{-27}$ erg s is the Planck constant and $k = 1.38 \times 10^{-16}$ erg K^{-1} is the Boltzmann constant.

Problem 6.4 Show that, in units of wavelength $\lambda = c/\nu$, the Planck function is

$$I_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}. \tag{6.21}$$

Problem 6.5 Use Gnuplot to make a plot of the Planck spectrum for temperatures T of $10, 10^3, 10^6$ and 10^9 degrees Kelvin. Use log Intensity (I_ν) versus log frequency (ν) for the axes.

For $h\nu/kT \ll 1$, the photons behave like classical particles. In this case

$$e^{h\nu/kT} \approx 1 + \frac{h\nu}{kT} + \ \ldots \tag{6.22}$$

and hence to first order

$$I_\nu = \frac{2\nu^2}{c^2}kT \qquad h\nu \ll kT \tag{6.23}$$

This is called the **Rayleigh-Jeans** part of the black-body spectrum.

For $h\nu/kT \gg 1$, the quantum nature of the photons comes into play. We have

$$e^{h\nu/kT} - 1 \approx e^{h\nu/kT} \tag{6.24}$$

and thus

$$I_\nu = \frac{2h\nu^3}{c^2}e^{-h\nu/kT} \qquad h\nu \gg kT. \tag{6.25}$$

This part of the black-body spectrum follows the Wien Law, and is seen as a rapid fall-off in the intensity going toward higher frequencies.

The microwave background radiation, thought to be a relic of the Big Bang, is almost exactly black-body.

The energy density u_ν for black-body radiation is given by

$$u_\nu = \frac{4\pi}{c}I_\nu \tag{6.26}$$

and the pressure P_ν of the field (due to photon momentum) is

$$P_\nu = \frac{4\pi}{3c}I_\nu. \tag{6.27}$$

6.4.1 Total flux

The total flux integrated over all frequencies is

$$I = \int_0^\infty I_\nu d\nu = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \tag{6.28}$$

where $x = h\nu/kT$. Performing the integral leads to

$$I = \frac{2\pi^4 k^4}{15c^2h^3}T^4 = \frac{\sigma}{\pi}T^4 \tag{6.29}$$

and thus the important result emerges that the luminosity L of a body in thermal equilibrium (i.e. at constant temperature) is

$$L \propto T^4. \tag{6.30}$$

The constant of proportionality is called the Stefan-Boltzmann constant, $\sigma = 2\pi^5k^4/15c^2h^3$ and is 5.67×10^{-5} erg cm^{-2} K^{-4} s^{-1} in cgs units.

For sources which are emitting radiation which is close to a black-body, it is very useful to define the **effective temperature** $T_{\rm eff}$, where

$$F = \int_0^\infty \cos\theta I_\nu d\nu d\Omega = \sigma T_{\rm eff}^4. \tag{6.31}$$

One can think of $T_{\rm eff}$ as a "best fitting" black body curve to the actual spectrum.

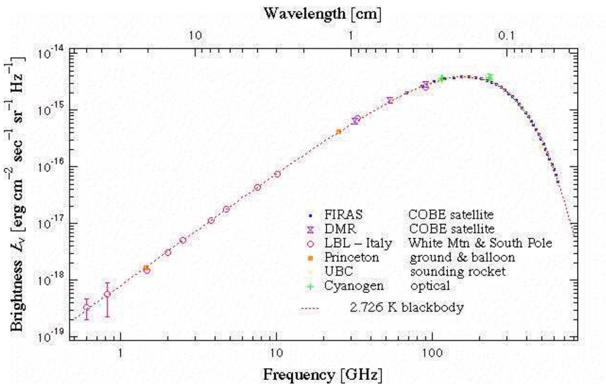

Figure 6.2. Brightness L_ν versus frequency ν of the Cosmic background radiation. The spectrum is very close to a perfect black-body, and has a temperature of of circa 2.73 Kelvin. Data points from the COBE FIRAS experiment match the theoretical curve so closely that the differences do not show up at this scale. Source : http://spectrum.lbl.gov/www/cobe/cobe.html

Problem 6.6 Use Gnuplot to make a plot of the Planck spectrum for temperatures T of $10, 10^2, 10^4, 10^6$ and 10^8 degrees Kelvin. Use log Intensity (I_ν) versus log frequency (ν) for the axes. Mark on the curves the Rayleigh-Jeans and Wien parts of the spectrum. Identify what type of radiation is associated with the peak of each spectrum (i.e. is it in the optical, X-ray, gamma-ray, radio, IR or UV region). Hint: Gnuplot may have trouble automatically scaling your plot axes to sensible values. You can use the command

plot [1E9:1E22] [1E-40:1E20]

to get useful limits on the axes, namely 1×10^9 to 1×10^{22} Hz for the x-axis and 1×10^{-40} to 1×10^{20} for the y-axis.

6.4.2 Monotonicity with temperature

For any two blackbody curves, the one with higher temperature has greater intensity than the cooler one for all wavelengths: i.e. a greater flux is emitted at all wavelengths.

Also, as $T \to 0$, the total flux $\to 0$,

and, as $T \to \infty$, the total flux $\to \infty$.

6.4.3 Wien displacement law

At what ν and λ are I_ν and I_λ maximum?

Problem 6.7 Find the peak luminosity for the functions I_ν and I_λ, by solving

$$\frac{dI_\nu}{d\nu} = 0 \quad \text{and} \quad \frac{\mathrm{d}I_\lambda}{\mathrm{d}\lambda} = 0$$

Useful substitutions to make are $x = \frac{h\nu}{kT}$ and $y = \frac{hc}{\lambda kT}$.

(Hint: the first case is equivalent to solving $x = 3(1 - e^{-x})$ and the second case to solving $y = 5(1 - e^{-y})$).

The problem above shows that the maximum emission in the black body curve occurs at different places in λ and ν. Let's denote these $\lambda_{\rm max}$ and $\nu_{\rm max}$.

At temperature T, the following relations hold:

$$\nu_{\max}/T = 5.88 \times 10^{10}\mathrm{HzK}^{-1}$$

and

$$\lambda_{\max}T = 0.290\mathrm{cmK}.$$

In other words $\nu_{\max}$ is a linear function of temperature, T. This simple relationship is called the **Wien Law**.

Problem 6.8 Check from your plots that the peak of the blackbody spectrum shifts linearly with temperature, i.e. that it follows the Wien Law.

6.4.3.1 Effective temperature

For sources which are emitting radiation which is close to a black-body, it is very useful to define the **effective temperature** $T_{\rm eff}$, where

$$F = \int_0^\infty \cos\theta I_\nu d\nu d\Omega = \sigma T_{\rm eff}^4. \tag{6.32}$$

One can think of $T_{\rm eff}$ as a "best fitting" black body curve to the actual spectrum.

Problem 6.9 Estimate the wavelength where the peak emission occurs for the three spectra in the figure. What temperature does each source have? Make an estimate of the "spectral type" of the stars (i.e. OBAFGK or M) from its temperature (a table of spectral types and temperatures is in a previous lecture).

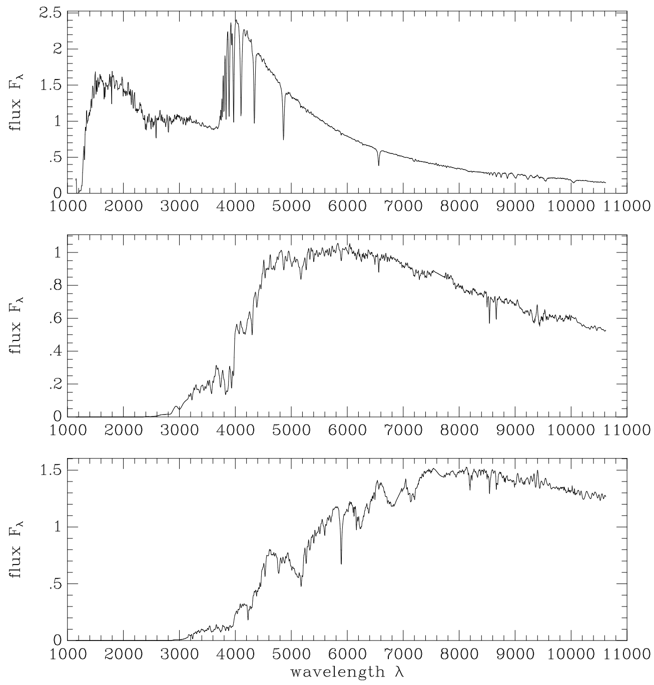

Figure 6.3. Spectra for three different stars (see problem 6.9).

6.5 MEAN OPACITY

In the deep stellar interior, the mean-free-path of photons is very short (see problem 6.2), and to a very good approximation the radiation field is isotropic and Planckian with a temperature which corresponds to the local gas temperature. Furthermore, the energy density in the general radiation field is much greater than the general outward energy flux. Let us assume the radiation field is black-body and dependant only on the temperature. The pressure is (as a function of temperature T)

$$P_\nu = \frac{4\pi I_\nu(T)}{3c} \tag{6.33}$$

where the outward flux is given by the pressure gradient (i.e. Eq. 6.19)

$$\pi F_\nu = -c\frac{dP_\nu}{d\tau_\nu}. \tag{6.34}$$

We can derive the total outward flux, F by integrating

$$\pi F = -\frac{4\pi}{3}\int_0^\infty \frac{1}{k_\nu}\frac{dI_\nu}{dT}\frac{dT}{ds}d\nu. \tag{6.35}$$

This integral suggests a definition of an *average opacity* $\bar{k}$, by

$$\frac{1}{\bar{k}} = \frac{\int_0^\infty \frac{1}{k_\nu}\frac{dI_\nu}{dT}d\nu}{\int_0^\infty \frac{dI_\nu}{dT}d\nu} \tag{6.36}$$

This quantity is called the **Rosseland mean opacity**.

Problem 6.10 Show that

$$\int_0^\infty \frac{dI_\nu}{dT}d\nu = \frac{4\sigma T^3}{\pi} \tag{6.37}$$

and hence that

$$\pi F = -\frac{4ac}{3\bar{k}}T^3\frac{dT}{ds} \tag{6.38}$$

where $\sigma = ac/4$.

In the context of a star, we are interested in the energy flow outward, and so we associate the path length s with the radius r. Consider the total radiation energy crossing a shell of radius r as having a luminosity $L(r)$. The area of this shell is $4\pi r^2$, and the flux crossing it is πF, so that

$$L = 4\pi r^2 \pi F = \frac{16\pi ac}{3\bar{k}}r^2T^3\frac{dT}{dr}. \tag{6.39}$$

It is often assumed in stellar interior calculations that the material is in **local thermodynamic equilibrium**. In LTE, the plasma is assumed to be at or very close to a constant temperature locally. The justification for this assumption is that the mean free path of the photons is typically very small, much smaller than the scale over which the temperature, pressure and density change in the star. The net outward flux of energy is the very small departure of the radiation field from isotropy. The radiation field locally can then be represented to very good precision as Planckian. Having established the intensity of the local radiation field and the temperature of the plasma, the pressure, density and distribution of atomic states may be determined.

The total rate of energy transfer outwards is broadly determined by the temperature gradient, rather than by interactions at specific frequencies, as shown by the luminosity equation (Eq 6.7). This is the reason that Rosseland was able to develop the mean opacity description above.

6.6 SOURCES OF OPACITY

We have developed equations to describe the flow of radiation, parameterising in terms of the opacity of the material. The absorption and emission of light by the material is ultimately determined by quantum mechanics of the ions or atoms of which it is composed. There are a variety of ways in which an atom can interact with a photon field. Broadly, opacity can be divided into **line-opacity** and **continuous opacity**. Line opacity is caused by the absorption of photons by atoms, such as occurs when an electron changes its energy level in the atom, and takes place at specific frequencies. In the outer layers of a star this process results in the familiar "lines" in the spectrum. Continuous opacity is of more interest inside the star. Because of the basic physical couplings between photons and ions, photons over a wide range of frequency are able to transfer energy/momentum to the stellar material, rather than at specific frequencies around lines. This is the origin of the term "continuous".

For some opacity processes, relatively simple analytical expressions for the frequency dependency can be obtained, in particular for protons, electrons and atoms with only one electron. For more complicated situations, one is forced to attempt detailed numerical calculations of the atomic states of the species, or make use of laboratory data (line lists).

The primary mechanisms which account for opacity are

- electron scattering
- free-free (or bremsstrahlung)
- bound-bound transitions
- bound-free (photoionisation and recombination)
- molecular transitions

In low temperature stars ($T < 5000$K) the main source of opacity is due to molecules (resonant lines or bands), Rayleigh scattering from neutral species, bound-free and free-free scattering off negative ions (e.g. H^-). For higher temperatures (up to 10^6 K) the main opacity source is bound-bound and bound-free absorption by atoms. At around 10^6 K, free-free absorption by electrons in the field of positive ions is the opacity source, while at higher temperatures still, the source is free electron scattering (Compton or Thomson scattering). We shall now look at these in more detail.

6.6.1 Electron scattering

When a photon collides with an electron initially at rest, the conservation of energy and angular momentum result in the electron being set into motion and a small wavelength change in the photon. For the case of photon energy much less than the rest mass energy of the electron (about 0.5

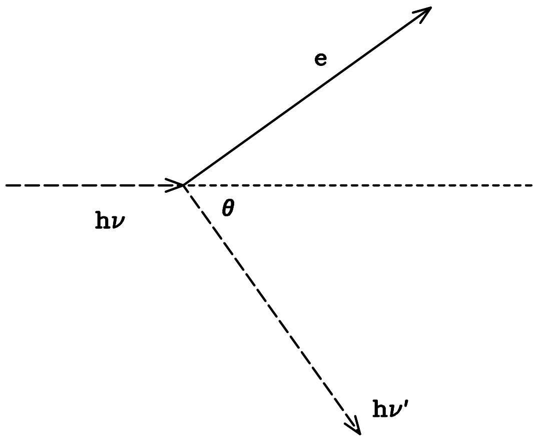

Figure 6.4. Electron scattering of a photon. A photon $h\nu$ collides with a stationary electron, which is ejected along the track marked e. The photon is scattered with a changed wavelength along $h\nu'$.

MeV which is in the X-ray region), the wavelength is increased and the electron gains kinetic energy. For higher energy photons, the wavelength is decreased.

For a photon scattered through an angle θ, there is a change in wavelength of $2\lambda_c \sin^2(\theta/2)$, where $\lambda_c = h/m_e c = 0.024$Åis the **Compton wavelength**. In the fully relativistic treatment this is known as **Compton scattering**. The non-relativistic case is called **Thomson scattering**. A derivation of the cross-section for Thomson scattering σ_T is beyond the scope of this course, but it can be derived from classical arguments which lead to an estimate of the classical electron radius, $r_e = e^2/m_e c^2$. The Thomson cross-section is

$$\sigma_T = \frac{8\pi}{3} r_e^2 = 6.65 \times 10^{-25} \mathrm{cm}^2. \tag{6.40}$$

The optical depth τ for Thomson scattering for an ionised gas is

$$\tau = n_e \sigma_T \tag{6.41}$$

where n_e is the electron density.

Problem 6.11 Find the optical depth for Thomson scattering for the following systems:

- The Sun, assuming it is a sphere of ionised Hydrogen with a *uniform density*. The mass of the Sun is $M_\odot = 2 \times 10^{33}$ g, and its radius is 7×10^5 km.
- Intergalactic space, over the distance of the Hubble radius, assuming that the electron density is 10^{-5} cm^{-3}. The Universe is about 15 billion years old and the Hubble radius is the distance light travels in this time.
- The disk of the Galaxy, for an average electron density of 0.1 cm^{-3}. The disk of the galaxy is about 300 pc thick and 20 kpc in radius, where 1 pc = 3.26 light years and 1 kpc = 1000 pc.

Discuss the results in terms of the physical sizes of the systems, i.e. which is optically thick and which optically thin.

In the case that a photon scatters off an electron which is part of an atom, a similiar analysis yields the **Raleigh**

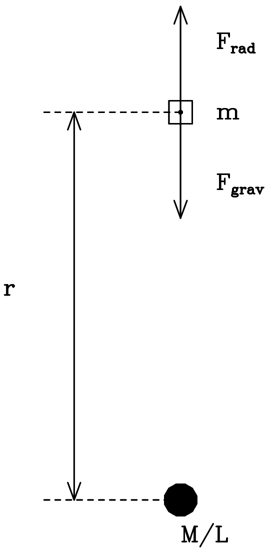

Figure 6.5. A small mass element m a distance r from a luminous body of mass to luminosity ratio M/L experiences an outward force due to radiation pressure, $F_{\rm rad}$ and an inward force due to gravity $F_{\rm grav}$. The Eddington limit is the condition that these forces balance, and sets an upper limit to the luminosity L of a body for a given mass, M.

scattering cross-section, σ_R, and applies when the photon has an energy less than the energy spacings of the atom, ΔE. Electrons are set into resonant motion along their orbits by the photon, and later re-emits a photon with the same energy (i.e. no change of wavelength) as the first photon. The scattering cross-section is related to the Thomson cross-section by

$$\sigma = \sigma_T \left[1 - (\nu_0/\nu)^2\right]^{-2} \tag{6.42}$$

where $h\nu$ is the photon energy and $h\nu_0$ is a measure of the restoring force during the harmonic oscillation. When the photon energy is small ($\nu \ll \nu_0$) we obtain the Raleigh cross-section

$$\sigma_R = \sigma_T (\nu/\nu_0)^4. \tag{6.43}$$

The assumption of small photon energy is equivalent to $kT \ll \Delta E$, and is valid for temperatures of up to a few 10^3 K. This process is only important in stellar atmospheres, particularly for cool (late-K and M) stars.

6.6.1.1 Eddington Limit

The Thomson cross-section may be used to yield an important limit on the luminosity of a self gravitating body, called the Eddington Limit.

Consider an element of mass m of a cloud of ionised gas, near a luminous object such as a hot star, with a mass to luminosity ratio of (M/L) (see figure 6.4). The radiation force outward on the cloud is

$$F_{\rm rad} = m \frac{\kappa L}{4\pi r^2 c} \tag{6.44}$$

where L is the luminosity of the central source at the cloud and κ is the mass absorption coefficient of the cloud, (i.e. the cross section per unit mass) and is defined by $k_\nu = \kappa_\nu \rho$.

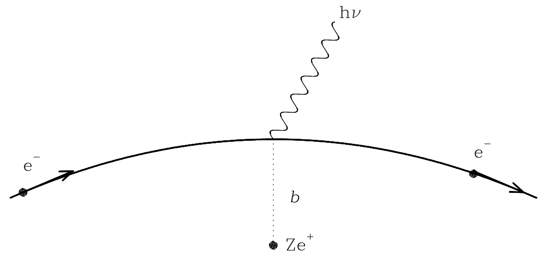

Figure 6.6. Free-free or Bremsstrahlung scattering. An electron e^- passes an ion with charge $Ze+$ with an impact parameter, b. Forces acting on the charge during the passage cause the emission or absorption of a photon, $h\nu$.

Problem 6.12 What is the force on the cloud due to the gravity of the central mass?
From outward and inward forces on the cloud show that if

$$\frac{M}{L} < \frac{\kappa}{4\pi G c} \tag{6.45}$$

then the cloud will be forced away by radiation pressure.

Problem 6.13 A minimum value for κ is that due to Thomson scattering off free electrons, where we consider the cloud to be fully ionised Hydrogen. Show the maximum luminosity that a central mass M can have while not ejecting matter spontaneously is

$$L_{\rm Edd} = \frac{4\pi G M c m_H}{\sigma_T} \tag{6.46}$$

where m_H is the mass of the Hydrogen atom. Hence, show that $L_{\rm Edd} = 1.25 \times 10^{38} (M/M_\odot)$ erg s^{-1}, where $M_\odot = 2 \times 10^{33}$ g. Using the mass and luminosity of the Sun, how close is the Sun to the Eddington limit?

6.6.2 Free-Free or Bremsstrahlung Scattering

When an electron moves through an electric field, it experiences an acceleration and radiates photons. In the particular case of free-free or Bremsstrahlung emission, the electron passes thorugh the Coulomb field of a positive ion, typically a proton. As the electron is accelerated during its approach to the ion it radiates photons, and then absorbs photons as it leaves the field of the ion. However, the conservation of momentum and energy mean that in the presence of the ion, these processes do not cancel out, and net absorption or emission takes place.

The acceleration acting on the electron as it passes by an ion of total charge Ze^+ is of order

$$\frac{Ze^2}{m_e b^2} \tag{6.47}$$

and this acceleration acts over a time scale of order $\Delta t = b/v$, where v is the velocity of the electron and b is the impact parameter of the collision (distance of closest approach) as shown in figure 6.5. The velocity change due to the acceleration is considered to be very small.

The radiated flux of an accelerated particle with charge e and acceleration dv/dt is given by

$$\frac{dE}{dt} = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{dv}{dt}\right)^2 \tag{6.48}$$

and thus the total emitted energy is

$$E_{\text{abs}} = \int_{-\infty}^{\infty} \frac{dE}{dt}dt = \frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{\infty}\left(\frac{dv}{dt}\right)^2 dt \tag{6.49}$$

Substituting the acceleration $dv/dt = Ze^2/m_e b^2$,

$$E_{\text{abs}} = \frac{2}{3}\left(\frac{Z^2e^6}{m_e b^2}\right)^2 \frac{b}{v} = \frac{2}{3}\frac{Z^2e^6}{m_e^2c^3}\frac{1}{b^3v}. \tag{6.50}$$

We can also compute the spectrum of emitted radiation, E_ν. As a Fourier series, the emitted flux takes the form

$$\frac{dE(\nu)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dE(t)}{dt}e^{-2\pi i\nu t}dt. \tag{6.51}$$

This integral will have non-negligible terms under the condition that $2\pi\nu t \approx 1$ during the time interval of the interaction b/v. Hence, the primary energy emission occurs around the frequency $\nu = v/2b\pi$.

The energy emitted by the electron as Bremsstrahlung radiation can just as well be absorbed by the electron in the field of the ion. This reverse process is then **Bremsstrahlung absorption**, for which we define a coefficient a_ν, being the absorption (from the surrounding radiation field) per ion per electron of velocity v.

The usual case of interest is that of thermodynamic equilibrium, in which the electron velocities are **Maxwellian**, i.e. they have a characteristic distribution $f(v)$ which depends only on the temperature T

$$f(v) = 4\pi\left(\frac{m_e}{2\pi kT}\right)^{\frac{3}{2}} v^2 \exp\left(-\frac{m_e v^2}{2kT}\right) \tag{6.52}$$

Problem 6.14 Use gnuplot to plot the distribution of velocities $f(v)$ as a function of v in electron gases of temperature $T = 10^3$ and 10^6 K. What is a typical electron velocity at the center of the Sun, where $T \approx 15 \times 10^6$ K?

In thermodynamic equilibrium, the energy density of the photon field (as shown in an earlier problem) is

$$u_\nu = \frac{4\pi}{c}I_\nu \tag{6.53}$$

so that the total energy absorbed by the electrons from the photon field is the product of the photon flux $cu_\nu d\nu$, the number density of ions, n_i, the number density of electrons, n_e, the fraction of electrons with velocity v, $f(v)dv$ and the absorption coefficient, a_ν. For completeness, we note that there is a further factor of $(1 - e^{-h\nu/kT})$, which takes into account the amount of stimulated emission (the bremsstrahlung re-emitted by the electron as a result of the changes produced during bremsstrahlung absorption).

Under thermal equilibrium, we have the final condition that the absorption and emission fluxes from the electrons must be equal (or their temperature would not be stable) and this allows us to determine the absorption coefficient

$$a_\nu = \frac{\pi}{3}\frac{Z^2e^6}{hcm_e^2 v\nu^3}. \tag{6.54}$$

Assuming that the velocities of the electrons $f(v)$ are Maxwellian, we can integrate the absorption coefficient over velocity to obtain the **thermal bremsstrahlung opacity** κ_{ff}. This can be shown to be of the form

$$\kappa_{\text{ff}} = \frac{4}{3}\left(\frac{2\pi}{m_e kT}\right)^{\frac{1}{2}}\frac{Z^2e^6}{hcm_e\nu^3}n_i n_e \bar{g}_{\text{ff}}(\nu) \tag{6.55}$$

where $\bar{g}_{\text{ff}}(\nu)$ is the **Gaunt factor** and is of order unity over a wide range of pressure and density conditions. In general its value has to be computed by numerical rather than analytical methods for the particular conditions under consideration.

Problem 6.15 The free-free opacity κ_{ff} has the general form

$$\kappa_{\text{ff}} \propto T^{-1/2}\nu^{-3} \tag{6.56}$$

Using the definition of the Rosseland mean opacity (Equation 6.36) show that

$$\bar{\kappa}_{\text{ff}} = \kappa_0 \rho T^{-7/2} \tag{6.57}$$

where κ_0 is a constant and a function of the composition of the star. (Hint: n_i and n_e are the number of ions and number of electrons in the ionised gas.)

This form of the opacity is known as **Kramer's opacity**.

The chemical composition of a star is generally parameterised in terms of the Hydrogen, Helium and heavy metal abundances *by mass*, X, Y and Z, where heavy metals are all those above Helium in the periodic table.

For stars like the Sun, $X \approx 0.75$, $Y \approx 0.23$ and $Z \approx 0.02$. The chemical composition of the gas after the Big Bang, and before stars began to form was of order $X \approx 0.6$ and $Y \approx 0.4$, most of the heavy elements having been created by stars subsequently.

Problem 6.16 Use these values of X, Y and Z to determine the relative fraction of Hydrogen and Helium *by number density* in the Sun and after the Big Bang. Assume that the heavy elements can be represented by Oxygen.

Taking into account the chemical composition of the star, the constant κ_0 may be written in the form

$$\kappa_0 = 3.9 \times 10^{22}(1 + X)(1 - Z)\bar{g}_{\text{ff}} \tag{6.58}$$

REFERENCES

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