Few-Body Problem: Theory and Computer Simulations
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Gravitational Scattering

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Abstract. We review some modern applications of the theory of few-body encounters between binaries and single stars. In particular we focus on the treatment of adiabatic encounters, in a regime which is of importance in encounters between a star and a planetary system in a star cluster.

1. Introduction: examples and applications

Roughly speaking, gravitational scattering will be defined as the study of few-body encounters in which particles interact by Newtonian gravity, with certain types of initial conditions; namely, a few (normally two or three) bound subsystems, such as single stars or binaries, approach from infinity. Then the problem is to characterise the outcome, usually in a statistical sense.

The new book by Valtonen & Karttunen (2006) will very quickly become the standard reference on this problem. While approximate analytical methods yield useful results in some limiting situations (see, for example, Sec.2.), computer simulation is an essential tool. And while efficient codes exist for computing individual scattering encounters (e.g. *triple*[1], *fewbody*[2]) the scattering packages in *starlab*[3] provide the additional, extremely valuable functionality of efficiently sampling parameter space. Many possibilities exist for graphic rendering of individual encounters, such as *GLanim*[4].

1.1. Examples and applications

Though the topic was developed throughout the twentieth century, it remains topical, because of fresh applications, such as the following:

Scattering of normal stars by a binary black hole As a result of mergers, central binary black holes are expected in many galaxies. Observationally, binary black holes are studied at high energies (e.g. NGC 6240, studied with Chandra: Komossa et al 2003) and at visual wavelengths (e.g. the famous Tuorla object OJ287: Sillanpää et al 1988). A problem of long standing is the evolution of

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[1] `http://www.ast.cam.ac.uk/~sverre/web/pages/nbody.htm`

[2] `http://www.astro.northwestern.edu/~fregeau/code/fewbody/`

[3] `http://www.ids.ias.edu/~starlab/`

[4] `http://grape.astron.s.u-tokyo.ac.jp/~makino/softwares/GLanim/`

the relative orbit of the black holes, as they scatter stars from the surrounding galaxy (Hills 1983, Gould 1991b, Mikkola & Valtonen 1992, Fukushige et al 1992, Quinlan 1996, Zier & Biermann 2001, Merritt 2001, 2002, Yu & Tremaine 2003, Milosavljević & Merritt 2003, Chatterjee et al 2003, Gualandris et al 2005).

Evolution of planetary systems in star clusters For the solar system, though it is not now in a star cluster, the question of stellar perturbations has been considered for a long time (e.g. Lyttleton & Yabushita 1965). In recent years this question has arisen because of the initially surprising absence of planets (searched for photometrically) in star clusters (e.g. the globular star cluster 47 Tuc; see Gilliland et al 2000, Weldrake et al 2005). From the theoretical point of view the study of such scattering is simplified by the fact that one component of the participating binary is effectively massless (Davies & Sigurdsson 2001, Spurzem et al 2003, Fregeau et al 2006). We shall say more about this in Sec. 2.

Capture of exotic particles by multiple systems The idea here is that the flux caused by interactions with normal matter may be enhanced in situations where the particles can be trapped gravitationally, which allows them multiple opportunities of interacting. This is a more speculative problem, but has still attracted considerable interest (Press & Spergel 1985, Gould 1987, 1988, 1991a, Damour & Krauss 1998, 1999, Gould & Alam 2001, Lundberg & Edsjö 2004). Theoretically it is the case where the incoming particle is nearly massless. The complexity of the problem is illustrated by a comparable problem of solar system dynamics: what fraction of comets and asteroids are destroyed by colliding with the sun? Several planetary resonances are involved in the fact that almost half of a sample of near-Earth asteroids end by colliding with the sun (Farinella et al 1994). It is also a common fate of short-period comets (Levison & Duncan 1994).

The M4 triple The nearby globular cluster M4 contains a triple system in which a distant companion, of mass comparable with or somewhat larger than Jupiter's mass, is in orbit about a binary consisting of a white dwarf and a millisecond pulsar. It is likely to have formed in a four-body scattering encounter between two binaries (Rasio et al 1995, Ford et al 2000, Fregeau et al 2006).

2. Adiabatic encounters

Consider a three-body scattering event involving a binary, with components of mass m_1, m_2, and a third single star of mass m_3. In this section we shall be concerned with situations where the encounter is both *tidal* and *adiabatic*. Let the (initial) semi-major axis of the binary be a, and v the typical relative velocity of the components. Then we shall estimate $v^2 \sim GM_{12}/a$, as in a circular orbit, where $M_{12} = m_1 + m_2$. Let the relative speed of the star and the binary, when far apart, be $V_{\rm inf}$, and let q be the distance of closest approach between them. Taking a Keplerian approximation for their relative motion, we see that, at close approach, their relative speed is $V^2 = V_{\rm inf}^2 + \dfrac{2GM_{123}}{q}$, where $M_{123} = m_1 + m_2 + m_3$, while the eccentricity of their relative motion is

$e' = 1 + \dfrac{qV_{\rm inf}^2}{GM_{123}}$. By plotting the curves $e' = 2$, $V/q = v/a$ (using the foregoing estimates) and $q = a$ we distinguish those encounters which are near-parabolic, adiabatic and tidal, respectively (Fig.1).

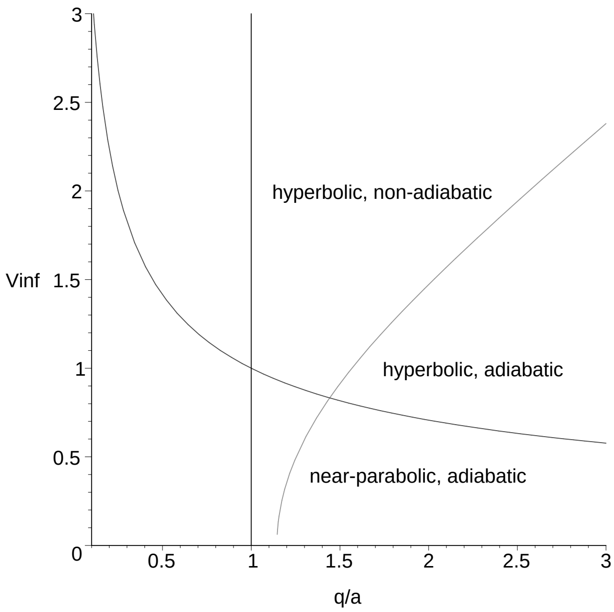

Figure 1. Regimes of tidal, adiabatic and hyperbolic encounters. Encounters to the right of the vertical line are tidal. These curves have been sketched for the case of equal masses, but do not depend sensitively on the masses unless m_3 is very large. $V_{\rm inf}$ is plotted in units of $\sqrt{GM_{123}/a}$.

Scattering problems like this can be approached analytically in various limiting regimes. High up in the diagram are non-adiabatic, impulsive encounters, which cover most situations involving "soft" binaries. At the bottom, just above the horizontal axis, in the tidal regime, are near-parabolic, adiabatic encounters. A theoretical study of encounters in this regime was carried out by Roy & Haddow (2003), who gave explicit formulae for the change in energy of the binary during the encounter. Heggie & Rasio (1996) had done a similar job for the change in eccentricity of the binary, and had actually extended their result to the regime of hyperbolic, adiabatic encounters. This regime is important for one of the problems mentioned earlier (Sec.1.1.), viz. the evolution of planetary systems in star clusters, where typical values of $V_{\rm inf}$ are comparable with the orbital velocity of a planet at a few AU. The main purpose of the present section is to extend the principal result of Roy & Haddow to this regime.

Using first-order perturbation theory, and truncating the perturbing potential between the binary and m_3 at quadrupole order, Roy & Haddow show that the change in energy of the binary is

$$\delta\varepsilon = -\frac{Gm_1m_2m_3}{M_{12}}\int_{-\infty}^{\infty}\dot{\mathbf{R}}.\frac{\partial\mathcal{R}}{\partial\mathbf{R}}dt, \tag{1}$$

where t is time, $\mathbf{R}$ is the position vector of m_3 relative to the barycentre of the binary, and $\mathcal{R}$ is the perturbing function. In turn, this can be taken to be of the form

$$\mathcal{R} = \frac{1}{R^5}\left\{\left[\frac{3}{2}e_1a^2(\hat{\mathbf{a}}.\mathbf{R})^2 - \frac{3}{2}e_2b^2(\hat{\mathbf{b}}.\mathbf{R})^2 - \frac{1}{2}ee_3a^2R^2\right]\cos M + 3e_4ab\hat{\mathbf{a}}.\mathbf{R}\hat{\mathbf{b}}.\mathbf{R}\sin M\right\}, \tag{2}$$

where a, b, e are, respectively, the semi-major axis, the semi-minor axis and eccentricity of the binary, $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ are unit vectors along the axes of the orbit of the binary, M is the mean anomaly of the binary, and the remaining coefficients are defined to be

$$\begin{aligned}
e_1 &= J_{-1}(e) - 2eJ_0(e) + 2eJ_2(e) - J_3(e)\\
e_2 &= J_{-1}(e) - J_3(e)\\
e_3 &= eJ_{-1}(e) - 2J_0(e) + 2J_2(e) - eJ_3(e)\\
e_4 &= J_{-1}(e) - eJ_0(e) - eJ_2(e) + J_3(e),
\end{aligned}$$

in which J_n is the Bessel function of the first kind of order n.

Roy & Haddow proceed to use formulae for parabolic motion for $\mathbf{R}$. We follow their same procedure using formulae for hyperbolic motion (e.g. Plummer 1918), i.e.

$$\begin{aligned}
\mathbf{R} &= a'(e' - \cosh F)\hat{\mathbf{A}} + b'\sinh F\hat{\mathbf{B}} && (3)\\
R &= a'(e'\cosh F - 1), && (4)
\end{aligned}$$

where $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ are unit vectors aligned with the axes of the hyperbolic relative orbit of m_3, a', b', e' are the hyperbolic analogues of a, b, e for this orbit, and F is the eccentric anomaly, related to time by

$$n't = e'\sinh F - F, \tag{5}$$

in which

$$n'^2a'^3 = GM_{123} \tag{6}$$

and we have assumed $t = 0$ at the time of closest approach. We also have $\cos M = \Re\exp(in(t - t_0))$, where t_0 is the time of pericentric passage in the binary, and $n^2a^3 = GM_{12}$.

From Eqs.(1) and (2), we see that a typical term in the integrand of $\delta\varepsilon$ is

$$\dot{\mathbf{R}}\cdot\frac{\partial}{\partial\mathbf{R}}\left(\frac{(\hat{\mathbf{a}}\cdot\mathbf{R})^2}{R^5}\right)\cos M = \Re\left(\frac{2\hat{\mathbf{a}}\cdot\mathbf{R}\hat{\mathbf{a}}\cdot\mathbf{R}^*R - 5(\hat{\mathbf{a}}\cdot\mathbf{R})^2R^*}{R^6}\dot{F}\exp(in(t - t_0))\right), \tag{7}$$

where * denotes differentiation with respect to F. Indeed all the terms in Eq.(1) can be expressed in terms of the integral

$$\mathcal{I} = \int_{-\infty}^{\infty}\frac{f(F)\dot{F}}{R^6}\exp(in(t - t_0))dt,$$

where f is a polynomial in $\cosh F$ and $\sinh F$. By means of Eq.(5), we then have

$$\mathcal{I} = \exp(-int_0) \int_{-\infty}^{\infty} \frac{f(F)}{R^6} \exp\left\{ i \frac{n(e' \sinh F - F)}{n'} \right\} dF. \tag{8}$$

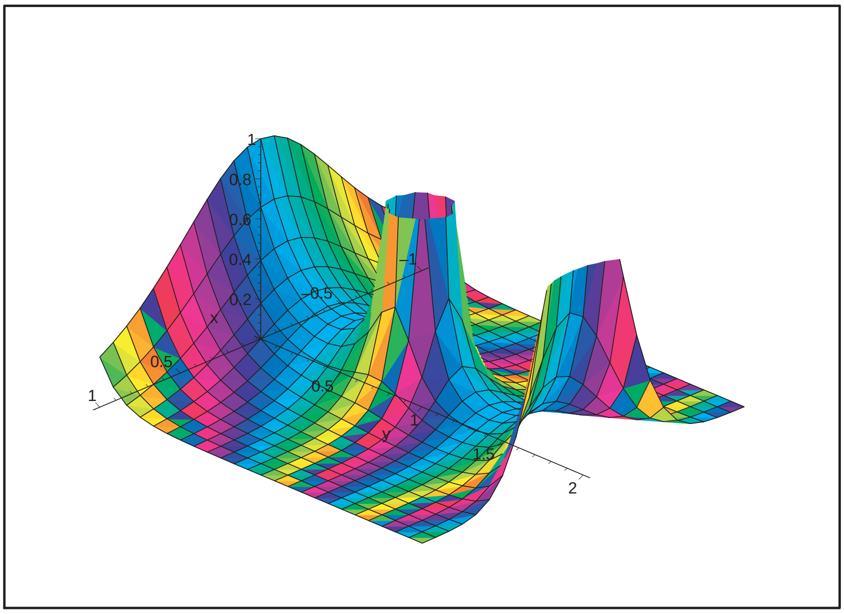

Figure 2. The modulus of the integrand of Eq.(8) in the region $-1 < \Re F < 1, 0 < \Im F < 2$ of the complex F-plane, for the case $e' \simeq 1$. Essentially the shape is that of a saddle, interrupted by a pole near $F = i$.

We handle this integral in the same way as in Heggie & Rasio (1996). Basically the method is steepest descents, but at the location of the saddle of the exponent, it turns out that $R = 0$. (This is responsible for the pole at about $F = i$ in Fig.(2).) First, therefore, we clear off the factors of R in the denominator. We deform the contour off the real F-axis to avoid the singularity at $F = 0$ in the following equivalent expression:

$$\mathcal{I} = \exp(-int_0) \int_{-\infty}^{\infty} \frac{f(F) R^*}{a'e' \sinh F R^6} \exp\left\{ i \frac{n(e' \sinh F - F)}{n'} \right\} dF,$$

where we have used the derivative of Eq.(4). Integrating by parts we can convert R^*/R^6 into $1/(5R^5)$, and in differentiating the remainder of the integrand we can treat everything except the exponential as constant: the adiabatic assumption implies that $n/n' \gg 1$, and so the derivative of the exponential factor dominates. Thus we have approximately

$$\mathcal{I} = \exp(-int_0) \frac{1}{5} \frac{in}{n'a'^2e'} \int_{-\infty}^{\infty} \frac{f(F)}{\sinh F} \frac{1}{R^4} \exp\left\{ i \frac{n(e' \sinh F - F)}{n'} \right\} dF,$$

where we have made use of Eq.(4) again. Doing this twice more gives the approximate result

$$\mathcal{I} = \exp(-int_0)\frac{1}{1.3.5}\left(\frac{in}{n'a'^2e'}\right)^3\int_{-\infty}^{\infty}\frac{f(F)}{\sinh^3 F}\exp\left\{i\frac{n(e'\sinh F - F)}{n'}\right\}dF. \quad (9)$$

Now we deform the contour onto the saddle of the exponent, which occurs where its derivative vanishes, i.e. $e'\cosh F - 1 = 0$. We choose the root $F_0 = i\arccos(1/e')$, which leads to

$$\sinh F_0 = i\sqrt{1-\frac{1}{e'^2}}, \quad \cosh F_0 = \frac{1}{e'}. \quad (10)$$

A quadratic approximation of the exponent around $F = F_0$ gives

$$i\frac{n}{n'}(e'\sinh F - F) \simeq -\frac{n}{n'}\left\{\sqrt{e'^2-1} - \arccos(1/e') + \frac{1}{2}\sqrt{e'^2-1}(F-F_0)^2\right\}.$$

Now the integral in Eq.(9) is easy, and gives

$$\begin{aligned}\mathcal{I} &= \exp(-int_0)\frac{1}{1.3.5}\left(\frac{in}{n'a'^2e'}\right)^3\sqrt{\frac{2\pi n'}{n}}(e'^2-1)^{-1/4}\frac{f(F_0)}{\sinh^3 F_0}\times\\ &\quad\times\exp\left(-\frac{n}{n'}(\sqrt{e'^2-1}-\arccos(1/e'))\right).\end{aligned}$$

Since $f(F)$ is defined implicitly in terms of $\mathbf{R}, R$ and their derivatives (cf. Eq.(7)), it is helpful to know that, when $F = F_0$, we have $\mathbf{R} = a'(e' - 1/e')(\hat{\mathbf{A}} + i\hat{\mathbf{B}}), R = 0, \mathbf{R}^* = a'\sqrt{1-1/e'^2}(-i\hat{\mathbf{A}} + \hat{\mathbf{B}})$ and $R^* = ia'\sqrt{e'^2-1}$, where we have used Eqs.(3),(4) and (10), and the fact that $b' = a'\sqrt{e'^2-1}$. (The fact that $R = 0$ at the saddle is the reason for the three integrations by parts.)

Now we see that the integral (with respect to t) of Eq.(7) is approximately

$$\begin{aligned}&\Re\exp(-int_0)\frac{i\sqrt{2\pi}(e'^2-1)^{3/4}}{15a'^3e'^2}\left(\frac{n}{n'}\right)^{5/2}(-5)(\hat{\mathbf{a}}.\hat{\mathbf{A}} + i\hat{\mathbf{a}}.\hat{\mathbf{B}})^2\times\\ &\quad\times\exp\left(-\frac{n}{n'}(\sqrt{e'^2-1}-\arccos(1/e'))\right).\end{aligned}$$

We use Eq.(6) and the corresponding relation for the binary to eliminate n and n', and eliminate a' in favour of the distance of closest approach, q. Thus the contribution of this term to $\delta\varepsilon$ in Eq.(1) is

$$\begin{aligned}&-\frac{Gm_1m_2m_3M_{12}^{1/4}}{M_{123}^{5/4}}\frac{\sqrt{2\pi}}{10}\frac{(e'+1)^{3/4}}{e'^2}\frac{q^{3/4}}{a^{7/4}}(-5e_1)\left\{((\hat{\mathbf{a}}.\hat{\mathbf{A}})^2 - (\hat{\mathbf{a}}.\hat{\mathbf{B}})^2)\sin nt_0 - \right.\\ &\quad\left.-2\hat{\mathbf{a}}.\hat{\mathbf{A}}\hat{\mathbf{a}}.\hat{\mathbf{B}}\cos nt_0\right\}\exp\left(-\left(\frac{M_{12}q^3}{M_{123}a^3}\right)^{1/2}\frac{\sqrt{e'^2-1}-\arccos(1/e')}{(e'-1)^{3/2}}\right).\end{aligned}$$

For ease of comparison, and a little economy, we introduce $K = \sqrt{\dfrac{2M_{12}q^3}{M_{123}a^3}}$, and so this contribution to $\delta\varepsilon$ is

$$-\frac{Gm_1m_2m_3}{M_{12}q^3}\frac{\sqrt{\pi}}{10}\frac{(e'+1)^{3/4}}{2^{3/4}e'^2}K^{5/2}(-5e_1a^2)\Big\{((\hat{\mathbf{a}}.\hat{\mathbf{A}})^2-(\hat{\mathbf{a}}.\hat{\mathbf{B}})^2)\sin nt_0- \\ -2\hat{\mathbf{a}}.\hat{\mathbf{A}}\hat{\mathbf{a}}.\hat{\mathbf{B}}\cos nt_0\Big\}\exp\left(-\frac{K}{\sqrt{2}}\frac{\sqrt{e'^2-1}-\arccos(1/e')}{(e'-1)^{3/2}}\right).$$

This is identical with the result of the corresponding terms in Roy & Haddow (the unnumbered equation below their Eq.(17)), except for the factor involving e', and the exponent. In the limit $e'\to 1$ the results are in agreement.

We can obtain a more convenient final expression by making the same changes to Eq.(19) in Roy & Haddow, yielding

$$\begin{aligned}\delta\varepsilon \;=\; & -\frac{Gm_1m_2m_3}{M_{12}q^3}\frac{\sqrt{\pi}}{8}\frac{(e'+1)^{3/4}}{2^{3/4}e'^2}K^{5/2}\exp\left(-\frac{K}{\sqrt{2}}\frac{\sqrt{e'^2-1}-\arccos(1/e')}{(e'-1)^{3/2}}\right)\times \\ & \times\Big\{e_1a^2[\sin(2\omega+nt_0)(\cos 2i-1)-\sin(2\omega+nt_0)\cos 2i\cos 2\Omega- \\ & -3\sin(2\omega+nt_0)\cos 2\Omega-4\sin 2\Omega\cos(2\omega+nt_0)\cos i]+ \\ & +e_2b^2[\sin(2\omega+nt_0)(1-\cos 2i)-\sin(2\omega+nt_0)\cos 2i\cos 2\Omega- \\ & -3\sin(2\omega+nt_0)\cos 2\Omega-4\cos(2\omega+nt_0)\sin 2\Omega\cos i]+ \\ & +e_4ab[-2\cos 2i\cos(2\omega+nt_0)\sin 2\Omega-6\cos(2\omega+nt_0)\sin 2\Omega- \\ & -8\cos 2\Omega\sin(2\omega+nt_0)\cos i]\Big\}, \end{aligned} \tag{11}$$

where ω, Ω and i describe the orientation of the path of m_3 in a frame aligned with the elliptical orbit of the binary. Thus Ω is the longitude of the ascending node, measured in the plane of the binary from its pericentre in the direction of its motion.

It would be desirable to test this formula numerically. So far the only tests to have been carried out involve qualitative comparison of scatter plots produced using both this formula and numerical scattering experiments. One reason why such a comparison is desirable is the cavalier nature of the perturbation calculation we have carried out. In particular the end result is exponentially small in the ratio a/q, whereas we have neglected terms which are only algebraically small (e.g. higher order terms omitted in Eq.(2)). Nevertheless the corresponding result for $e'=1$ works well (Roy & Haddow 2003). Finally we remark that Eq.(11) gives a null result when $e=0$. Roy & Haddow show how to deal with this when the encounter is parabolic, and it is expected that a result for the general case could be provided along the lines of the above calculation.

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