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An extension of the Free-Fall Problem

Kenji Kuwabara

Department of Applied Physics, Faculty of Science and Engineering
Waseda University, Tokyo 169-8555, Japan

Kiyotaka Tanikawa

National Astronomical Observatory, Mitaka, Tokyo 181-8588, Japan

Abstract. The planar three-body problem with zero initial velocities is systematically studied as the Free-Fall Problem (FFP). We define a new problem with non-zero initial velocity, as an extension of the FFP.

1. Introduction and motivation

The three-body problem with zero initial velocities is called the free-fall problem (FFP), and has been extensively studied by Russian and Japanese schools (Agekyan and Anosova, 1968, Anosova 1986, Tanikawa, 1995, Tanikawa, 2000). In the present report, we extend the problem to systematically include the initial velocities.

In the free-fall three-body problem, the initial value space is finite because of the similarity of the configuration, and we can study the property of the structure of the initial value space in this finite region. However, it is more desirable to put the problem into a wider scope, that is, the problem should be considered in the full, planar phase space. In full phase space, the FFP occupies a low dimensional manifold. We want to somehow 'blow up' this manifold by adding velocities to the problem.

The aim of our project is to investigate the structure of the equal mass planar three-body problem. The purpose of the present report is to make clear what is useful for our aim. Our attempt has been partially successful.

2. The Free-Fall Problem

The FFP is a planar problem. The total energy of the three bodies $m_i, i = 1, 2, 3$ is negative and their angular momentum is zero. We consider the equal mass case: $m_1 = m_2 = m_3$. In this problem, motions starting from similar triangles transform into one another under appropriate changes of coordinate and time, so we identify these motions. Dissimilar triangles correspond to independent motions.

Let mass points m_2 and m_3 start at rest at $A(-0.5, 0.0)$ and $B(+0.5, 0.0)$, respectively in the (x, y) plane and m_1 start at rest at a point $P(x, y)$ where

$$(x, y) \in \{(x, y) : x \geq 0, y \geq 0, (x + 0.5)^2 + y^2 \leq 1\}. \quad (1)$$

If m_1 changes position in D, then triangles satisfying the condition $\overline{AB} \geq \overline{PA} \geq \overline{PB}$ are exhausted. Conversely, any triangle is similar to one of the triangles formed by three mass points m_1,m_2,m_3 as above. Thus the positions of $P \in D$ specify all possible initial conditions in FFP.

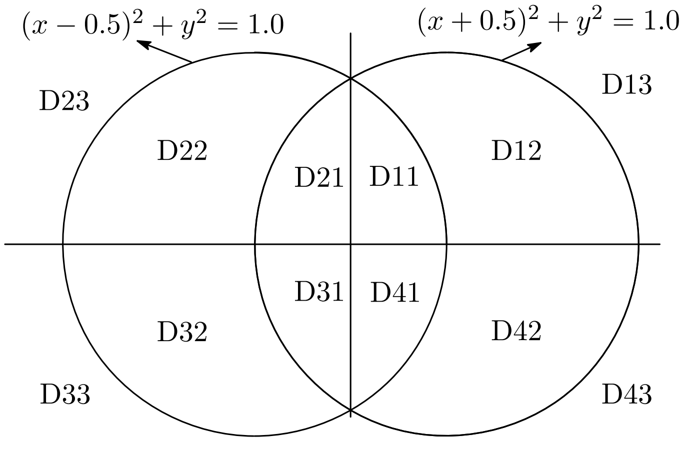
3. New initial condition 1 (angular momentum)

An attempt to include initial velocity was numerically carried out by Anosova, Bertov and Orlov (1984). One of their interesting observations is the angular momentum effect on the motion of triple systems. The angular momentum seems to make systems stable. A systematic study of angular momentum effects is important, and our project has started with such a motivation.

Anosova et al. (1984) considered the region D of the free-fall problem. They supposed that the system rotates in the plane of the initial triangle (2D problem) counterclockwise; the velocity vectors of components A (the distant component) and C (inside D) are orthogonal to their radius-vectors in the center-of-mass coordinate system; the angular momenta of these bodies are the same; the velocity of component B is given so that the center-of-mass of the triple system is motionless; the speed of rotation is parameterized by the initial virial ratio k. Thus, the initial conditions are defined by three parameters: coordinates (x, y) of the C component in region D and the virial ratio k.

However, their formulation loses the boundedness of the initial configuration space. This boundedness is one of the most important properties of the FFP. So the first thing we should do is recover the boundedness.

Figure 1. The initial map



Any given triangular configuration will lie in one of twelve regions (Fig.1) in the FFP. However if we give the triangle tips arbitrary velocities, the initial conditions may no longer be similar, and we must carefully choose our conditions. Our set of initial conditions is very similar to those of Anosova et al. (1984), except that velocities are given in a different way. The position of m_2 and m_3 are the same as in the FFP. We put m_1 at any place in the (x, y) plane. The conditions satisfied by the the initial velocities are:

1) each velocity vector is normal to the corresponding position vector from the gravity center,

2) the sum of the velocity vectors is zero,

3) the ratio between total kinetic energy and total potential energy is constant, i.e., the virial ratio is constant.

Following from the similarity transformation, we can classify twelve regions in the initial value space to two groups as follows.

1; $D_{11}, D_{13}, D_{22}, D_{31}, D_{33}, D_{42}$,
2; $D_{12}, D_{21}, D_{23}, D_{32}, D_{41}, D_{43}$.

Thus we need only consider D_{11} and D_{12} without loss of generality. In our definition, the initial condition space becomes compact and all the velocity vectors made by these conditions contribute to the angular momentum.

4. New initial condition 2 (time derivative of inertia moment)

There is another direction of extension of the FFP. The conditions introduced here are similar to the conditions of section §3. We only modify condition (1) of the former section to

1′) Each velocity vector is parallel to the corresponding position vector from the gravity center.

The classification of regions is the same as in the former case. This initial condition is independent of the one in §3, because all the velocity vectors contribute to the time derivative of the inertia moment, and the total angular moment is zero.

5. Discussion

In this report, we extend the Free-Fall Problem in two directions. The first is to give the system angular momentum, and the second is to give the system a time derivative of the inertia moment.

Now we discuss the direction of research to which we intend to proceed. The number of variables of the planar three-body problem is twelve. We know that there are 6 integrals of motion for the planar three-body problem: namely the four integrals of motion of the center of gravity, the integral of angular momentum, and the integral of energy. By using these quantities, we can reduce the degree of freedom of this system to three. Correspondingly we need to consider the six-dimensional phase space. It is very difficult to consider the whole structure of six-dimensional phase space, but when we set the masses to be equal, then from the results of this report, we find two independent directions in this six-dimensional phase space. For the moment, we are not successful in finding another independent direction. If we are to be successful, then we will have a five-dimensional surface of section comprising the (x, y) plane and three independent directions in the six-dimensional phase space.

Additionally, if the virial ratio is very large and the total energy of the system is of high positive value, then the type of final motion is rather simple in the initial value space. So to learn the qualitative behaviour, we may only consider finite values of the virial ratio. This is very useful for the computer simulations we are starting. The results of the simulations will be reported in the near future.

Finally we summarize our plans for future work;

1) To find the remaining independent direction in phase space;

2) To prove that the axis found makes the 'global' surface of section;

3) To study the precise structure in the five dimensional 'global' surface of section.

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