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The Rectilinear Three-body Problem using Symbolic Sequences II. Role of periodic orbits

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Abstract.
We have studied the behaviour of the periodic orbits in the rectilinear three-body system, bifurcated from the Schubart region, by varying the mass parameter. We find that the periodic points with rotation number $(n-2)/n$ dominate the structure on the Poincaré section.

1. Introduction

We study the structure on the Poincaré section in the rectilinear three-body system, continuing with our preceding work (Saito & Tanikawa, submitted; hereafter Paper I). In Paper I, we studied how the structures on the Poincaré section change as the central mass varies. The Poincaré section has the following hierarchical structure: the Schubart orbit, its stable region, an outer chaotic scattering region composed of arch shaped blocks, and a yet-more-distant region of fast escape. Although it had already been shown by Hietarinta and Mikkola (1993) that the number of arches increases as the central body becomes lighter, the improvement in our study was to reveal that a new arch is created by a lot of 'germ' shaped blocks, bifurcated from the arches using the symbolic sequence, which was applied for the equal mass case by Tanikawa and Mikkola (2000). The aim of this paper is to study how these 'germs' are related to the periodic orbit.

2. Method

2.1. Setting of the Problem

In the rectilinear three-body system, three particles run on a line and are mutually attracted by the Newtonian force. We denote the masses of the three particles by m_1, m_0, and m_2 (from the left), and the mutual distances $\overline{m_1 m_0}$ and $\overline{m_0 m_2}$ by q_1 and q_2. Taking (q_1, q_2) as coordinates and introducing their conjugate momenta (p_1, p_2), we have the Hamiltonian and the equation of the motion:

$$H = \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_0}\right)p_1^2 + \frac{1}{2}\left(\frac{1}{m_0}+\frac{1}{m_2}\right)p_2^2 - \frac{p_1 p_2}{m_0} - \frac{m_1 m_0}{q_1} - \frac{m_0 m_2}{q_2} - \frac{m_1 m_2}{q_1+q_2} \quad (1)$$

$$q_i = \partial H/\partial p_i, \quad p_i = -\partial H/\partial q_i \tag{2}$$

We consider the case $H(q,p) = E = -1 < 0$ and symmetric mass configuration, since we found that structure on the Poincaré section is less sensitive to asymmetry of the mass ratio. We express the mass ratio using the *mass parameter* a as $m_1 = m_2 = 1 - a$ and $m_0 = 1 + 2a$.

2.2. Poincaré Section

We use the Poincaré map as originally introduced by Mikkola and Hietarinta in order to study the structure in phase space (for the properties of this Poincaré map, see Hietarinta & Mikkola, 1993). This Poincaré section is defined by $q_1 = \Lambda q_2$ with the parameter Λ depending only on masses m_1, m_0 and m_2 (under the constraint $H(q,p) = -1$). We then introduce the coordinates (θ, R) defined by the following:

$$R = \tfrac{1}{2}(q_1 + q_2)|_{q_1=\Lambda q_2}, \quad \theta = \arctan\left(\frac{r_2\sqrt{\kappa}\dot{q}_1 - r_1\sqrt{\kappa}\dot{q}_2}{\sqrt{A-\kappa r_2{}^2}\dot{q}_1 - \sqrt{B-\kappa r_1{}^2}\dot{q}_2}\right), \tag{3}$$

where
$$0^\circ \leqq \theta \leqq 360^\circ, \ 0 \leqq R \leqq R_{\max},$$
$$A = \frac{m_1(m_0+m_2)}{2M}, \ B = \frac{m_2(m_0+m_1)}{2M}, \ C = \frac{m_2 m_0}{2M}, \ M = m_1 + m_0 + m_2.$$
$$r_1 = \frac{2\tau}{\tau+1}, \ r_2 = \frac{2}{\tau+1}, \ \kappa = \frac{4AB-C^2}{4(Ar_1{}^2+Br_2{}^2+Cr_1r_2)}.$$

The boundaries $R = 0, R_{\max}$ correspond to triple collision and the case $\dot{q}_1 = \dot{q}_2 = 0$, respectively. We divide the Poincaré section into the side Π with $\theta \leqq 180^\circ$ and the side Π^* with $\theta \geqq 180^\circ$. We then define the Poincaré map $x' = T(x)$ with $x, x' \in \Pi$, where the orbit integrated from x intersect at x' with Π at the next intersection.

2.3. Symbolic Sequence

Tanikawa and Hietarinta (2000) described the behaviour of particles using symbolic sequences instead of the orbit itself. A symbolic sequence is sequence of symbols corresponding to collisions experienced during the orbit. Here, the symbols are '0' (for a triple collision), '1' (for a collision between m_1 and m_0), or '2'(for a collision between m_0 and m_2).

In Paper I, we introduced *cylinders* (sets of symbolic sequences) for this symbol sequence. The definition of this cylinder is the following:

$$S_{c,j} \equiv \begin{cases} \{(21)^i(2)^j \cdots | i \geq 0, j \geq 1\} & (\text{if } c = 2i+1) \\ \{(21)^i(1)^j \cdots | i \geq 1, j \geq 1\} & (\text{if } c = 2i) \end{cases}, \quad S_c \equiv \cup_{j<\infty} S_{c,j}. \tag{4}$$

In particular, the *Schubart orbit* and its stable region has the symbol sequence $(21)^\infty \in S_\infty$. In Paper I, for the orbits starting from 500×300 grid points on Π, we calculate the symbolic sequence. We then partitioned Π into regions such that a region has a number c_0 and the symbolic sequences of all points in this region belong to S_{c_0}.

2.4. Rotation Number

The intersection Π of the Schubart orbit with Π is the fixed point of the Poincaré map T, namely $\boldsymbol{P}_0 = T(\boldsymbol{P}_0)$. The *rotation number* is a quantity which describes the rotational motion under T, when $\boldsymbol{P}_0$ is stable. First, we introduce polar coordinates (D, A) whose origin is at $\boldsymbol{P}_0 = (\theta_0, R_0)$:

$$D\cos A = \theta - \theta_0, \quad D\sin A = 100(R - R_0)/R_{\max}. \tag{5}$$

We then define the rotation number $\alpha_\infty(D, A)$ at (D, A) by the following equations:

$$\mathrm{diff}(A', A) = \begin{cases} A' - A & (\text{if } A' - A \geqq 0) \\ A' - A + 360° & (\text{if } A' - A < 0) \end{cases} \tag{6}$$

$$\alpha(D, A; n) = \frac{1}{360° n}\sum_{i=1}^{n} \mathrm{diff}(A^{(i)}, A^{(i-1)}), \quad \text{where } (D^{(i)}, A^{(i)}) = T^n(D, A) \tag{7}$$

$$\alpha_\infty(D, A) = \lim_{n\to\infty} \alpha(D, A; n) \tag{8}$$

The difference of A is always measured to the positive direction by $\mathrm{diff}(A', A)$ in Eq.(6). The definition in Eq.(8) shows that the rotation number is the average difference of A over an infinite number of mapping iterations. In actuality, since it cannot be calculated numerically for $(D, A) \neq (0, 0)$, we use an approximated rotation number $\alpha(D, A; n)$ for certain values of n, defined in Eq.(7). We here note that, when the mass parameter $a = a_0$ and $\boldsymbol{P} = (D, A)$, then if necessary, we write $\alpha(D, A; n)$ as $\alpha(\boldsymbol{P}; n)$ or $\alpha(D, A, a; n)$.

If $\boldsymbol{P}$ is periodic, $\alpha_\infty(\boldsymbol{P}) = q/p$ with integers p (period) and q, and $\alpha(\boldsymbol{P}; p) = \alpha_\infty(\boldsymbol{P})$. This property helps us to find the periodic points from the distribution of $\alpha(\boldsymbol{P}; n)$. Let us consider the limiting value of $\alpha_\infty(\boldsymbol{P})$ as $\boldsymbol{P} \to \boldsymbol{P}_0$, which is calculated from the eigenvalues of the linearised map of T at P_0. When this limit is rational, the fixed point and the periodic points are degenerate. If the mass parameter a is changed, the periodic points have finite distances to the fixed point or vanish.

3. Results

Here, we show the role of periodic points bifurcated from the fixed point on the structural change of the Poincaré section Π when the mass configuration is changed. First, we show the common properties shared by the periodic points over the whole range of the mass parameter a. We then follow the periodic points over a certain range of a corresponding to a value of α_∞, and show how these points are related in detail to the structures on Π.

3.1. Properties of the Periodic Points

Even though the periodic points bifurcate (or disappear) at arbitrary a that $\alpha_\infty(\boldsymbol{P}_0, a)$ is rational, most of them have low stability, and few of them have visible influence on the structure of Π. We have found that such dominant periodic points have rotation numbers with the following form:

$$\alpha_\infty = \frac{n-2}{n}, \text{ where } n = 3, 4, 5, \cdots, \text{ (and reduce the RHS by 2 for even } n). \tag{9}$$

The number of these points is always $2n$. If n is odd, bifurcated periodic orbits are composed of one stable and one unstable orbit with period n, whereas if n is even, there are two stable and two unstable orbits with period $n/2$.

The periodic points with α_∞ listed in Eq. 9 appear at $a = a_{n-2/n}$ where $\alpha(\boldsymbol{P}_0, a) = (n-2)/n$: for example, $a_{1/3} = 0.40$, $a_{1/2} = 0.0198$, $a_{3/5} = -0.15$, $a_{2/3} = -0.25$. Although we do not show it in a figure, $\alpha(\boldsymbol{P}_0, a)$ monotonically decreases from 1 to 0 (all possible values of the rotation number) as a increases from 1 to $-1/2$. The mass parameter $a = a_{1/3}$ is the critical point. The periodic points degenerate with P_0 at $a = a_{(n-2)/n}$ leave P_0, as a decreases if $a < a_{1/3}$, or as a increases if $a > a_{1/3}$.

3.2. The influence of the periodic points on the structure on Π

We shall follow the periodic points with $\alpha = 3/5$ with decreasing a. From our observation of the periodic points with $\alpha = (n-2)/n$ for $3 \leqq n \leqq 20$, we know their behaviour is basically similar. We here show our observation for the $\alpha = 3/5$ case. Figure 1 shows the periodic points on Π over the partitioned Π according to S_c. The colour and the printed number of each region is the value of c of S_c corresponding to the region. The central region corresponds to $\cup_{c \geqq 32}^{\infty} S_c$, which approximates the extent of the Schubart region corresponding to S_∞. At $a = -0.15$ in Fig.1(a), there are periodic points s_i and u_i ($i = 1, 2, 3, 4, 5$) with $\alpha_\infty = 3/5$ in the Schubart region. There are two periodic orbits, and the stable one intersects with $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, $\boldsymbol{s}_3$, $\boldsymbol{s}_4$, $\boldsymbol{s}_5$, and again $\boldsymbol{s}_1$, while the unstable one intersects with $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, $\boldsymbol{u}_4$, $\boldsymbol{u}_5$, and again $\boldsymbol{u}_1$, in that order. Approximate separatrices with pentagrammic shape are also shown. At $a = -0.16$ in Fig.1 (b), the periodic points are more distant from $\boldsymbol{P}$ than those at $a = -0.15$. The separatrices reach the front of the border of the Schubart region. There are blocks of regions with germ-like shape (hereafter we call them 'germs'), which grow along the separatrices. In this figure, these blocks have regions marked with one of the following: $\{5', 9', \cdots\}$, $\{9'', 13', \cdots\}$, $\{8', 12', \cdots\}$, and $\{7', 11', \cdots\}$. At $a = -0.166$, the germs gather around $\boldsymbol{s}_i$. The region corresponding to S_∞ is broken into the inner pentagon and five small triangular areas. They are the new Schubart region and the stable regions of $\boldsymbol{s}_i$. As a decreases, the germs gathering around s_i sink toward valleys between arches, whereas the stable regions of $\boldsymbol{s}_1$ shrink. We can obtain the coordinates of $\boldsymbol{s}_i$ until $a = -0.186$, solving $\boldsymbol{s}_i = T^5(\boldsymbol{s}_i)$. As can be seen in Fig.1(d), the extent of their stable regions is unresolved.

Let us consider the relation between the observations above and the structure we revealed in Paper I. The Poincaré section contains: a hierarchical structure of the Schubart orbit, its stable region (the Schubart region), arch shaped blocks surrounding this stable region, and further outside the fast escaping region. Their symbolic sequence belongs to S_∞, S_∞, S_c, and $S_{(c,\infty)}$. An arch shaped block corresponds to the union of the cylinders $\cup_{c \in C(r)} S_c$, where $C(r) = \{c | c \equiv r (\text{mod } n_{\text{arch}})\}$, the number of arches n_{arch}, and a unique number r given for the arch. Due to this composition rule, all the regions in an arch must be replaced, when the new arch appears. The germs bifurcating from the arches make this replacement. This summarises our understanding reported in Paper I. In the present study, we find that such germs are the structure associated with the stable periodic points s_i according to the observation for Fig.1(b). Moreover, in Fig.1(c), looking at one of s_i, its stable region, and the gathering germs, we can consider them as the hierarchical structure similar to that of the Schubart orbit.

References

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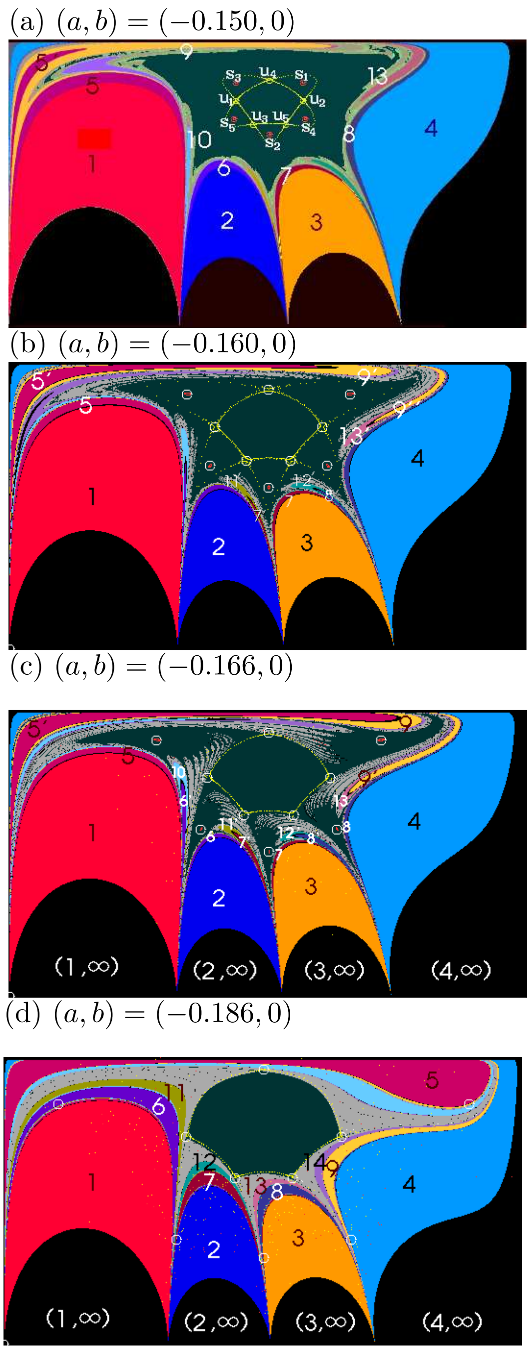

Figure 1. The periodic points with $\alpha = 3/5$ over the regions on Π