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Stability limit for hierarchical three-body systems

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Abstract.

We discuss the stability of a hierarchical three-body system. First we derive an analytical expression for the energy exchange between a binary and a third body in a single encounter. Then a stability limit for a single encounter is defined. Finally the stability limit for a triple system to survive for 10 000 revolutions of the outer orbit is calculated and refined using numerical orbit calculations.

1. Introduction

The three-body problem in gravitational physics has remained a central problem since the times of Isaac Newton. It can be simply stated as the problem of finding orbits of three bodies under their mutual inverse-square law gravitational attraction, starting from any given initial positions and velocities, and extending in time both backward and forward to infinity. The problem has been studied by a number of famous mathematicians, eg. Euler, Lagrange, Jacobi, Tisserand, Hill, Poincaré, Sundman, Burrau, Kozai, Kustaanheimo, Stiefel, Henon, Marchal and Szebehely. In what may be called the classical period, analytical methods dominated, but since around the mid-1960's the possibility of accurate orbit calculations in even the most general problems have brought a new dimension to the study of the three-body problem. One of the early important findings of the studies using powerful computers was the discovery of the inherent instability of three-body systems. Two of the bodies form a binary while the third body escapes. This process has been termed gravitational slingshot. One of the challenges today is to understand the principles of the gravitational slingshot as well as to establish stability boundaries in the phase space which separates unstable systems, leading to slingshot, from stable three-body systems. These investigations have numerous important applications in modern astrophysics. The derivations in this paper are found in detail in Valtonen and Karttunen (2005). We start by considering perturbations by a distant companion of a binary.

2. Perturbations of a and e

The Lagrangian equations are the traditional approach to perturbations. Here we will study another method for finding the effect of perturbations on the semi-major axis a and eccentricity e of a perturbed binary. This method can be applied to a great variety of perturbations.

We begin with the Newtonian equation of motion in the form

$$\ddot{\boldsymbol{r}} = -\mu \boldsymbol{r}/r^3 + \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{r}$ is the relative position vector, $\mu = G(m_a + m_b)$, m_a and m_b are the masses of the binary components, and $\boldsymbol{f}$ is a (small) extra term due to perturbations. It can be any vector-valued quantity with a dimension of acceleration, and it may be a function of the position and velocity.

First start by defining the angular momentum per unit mass $\boldsymbol{k}$ and a vector related to the eccentricity of the binary orbit $\boldsymbol{e}$:

$$\begin{aligned} \boldsymbol{k} &= \boldsymbol{r} \times \dot{\boldsymbol{r}}, \\ -\mu \boldsymbol{e} &= \boldsymbol{k} \times \dot{\boldsymbol{r}} + \mu \boldsymbol{r}/r. \end{aligned} \tag{2}$$

The time derivatives of these are

$$\dot{\boldsymbol{k}} = \boldsymbol{r} \times \boldsymbol{f}, \tag{3}$$

$$-\mu \dot{\boldsymbol{e}} = \dot{\boldsymbol{k}} \times \dot{\boldsymbol{r}} + \boldsymbol{k} \times \boldsymbol{f}. \tag{4}$$

We know that $\boldsymbol{k}$ is perpendicular to the orbital plane and its length depends on the parameter p of the orbit. Therefore it can be expressed in terms of p and a unit vector $\hat{\boldsymbol{e}}_\zeta$, perpendicular to the orbital plane:

$$\boldsymbol{k} = \sqrt{p\mu}\,\hat{\boldsymbol{e}}_\zeta. \tag{5}$$

Similarly, the length of $\boldsymbol{e}$ is the eccentricity of the orbit, and its direction is the direction of the perihelion.

$$\boldsymbol{e} = e\hat{\boldsymbol{e}}_\xi. \tag{6}$$

Using these we find the following expressions for the derivatives of $\boldsymbol{k}$ and $\boldsymbol{e}$:

$$\dot{\boldsymbol{k}} = \frac{1}{2}\sqrt{\mu/p}\,\dot{p}\hat{\boldsymbol{e}}_\zeta + \sqrt{p\mu}\,\dot{\hat{\boldsymbol{e}}}_\zeta, \tag{7}$$

$$\dot{\boldsymbol{e}} = \dot{e}\hat{\boldsymbol{e}}_\xi + e\dot{\hat{\boldsymbol{e}}}_\xi. \tag{8}$$

Now we may find the change in the eccentricity. Since $\hat{\boldsymbol{e}}_\xi \cdot \dot{\hat{e}}_\xi = 0$, the scalar product of the previous equation with $\hat{\boldsymbol{e}}_\xi$ gives

$$\hat{\boldsymbol{e}}_\xi \cdot \dot{\boldsymbol{e}} = \dot{e}, \tag{9}$$

from which

$$\dot{e} = \frac{1}{\mu}(\hat{\boldsymbol{e}}_\xi \cdot \dot{\boldsymbol{r}} \times (\boldsymbol{r} \times \boldsymbol{f}) + \sqrt{p\mu}\hat{\boldsymbol{e}}_\eta \cdot \boldsymbol{f}). \tag{10}$$

The change in the semi-latus rectum of the orbit is found by taking the scalar product of Eq. (7) and $\hat{\boldsymbol{e}}_\zeta$:

$$\hat{\boldsymbol{e}}_\zeta \cdot \dot{\boldsymbol{k}} = \frac{1}{2}\sqrt{\mu/p}\,\dot{p},$$

from which

$$\dot{p} = 2\sqrt{p/\mu}\,\hat{\boldsymbol{e}}_\zeta \cdot (\boldsymbol{r} \times \boldsymbol{f}). \tag{11}$$

The effect on the semi-major axis is found from the equation $a = p/(1-e^2)$, when we know the changes in p and e:

$$\dot{a} = \frac{\dot{p}}{1-e^2} + \frac{2pe\dot{e}}{(1-e^2)^2}. \tag{12}$$

One may also derive $\dot{a}$ directly by applying the method of perturbative differentiation to a Keplerian orbit.

3. Binary evolution with a constant perturbing force

As an application of the previous section we consider first order secular perturbations of the semi-major axis of the inner binary, under a constant perturbing force. This is relevant to highly hierarchical binaries where the ratio of the semi-major axes of outer and inner binaries a_e/a_i is large. Then the outer binary component appears practically stationary in relation to the fast orbital motion of the inner binary. We will find that there is no secular energy exchange between the inner and outer binary in this situation.

The first order perturbing acceleration is

$$\boldsymbol{f} = -\frac{Gm_3}{R_3^3}\left(\boldsymbol{r} - \frac{3\boldsymbol{r}\cdot\boldsymbol{R}_3}{R_3^2}\boldsymbol{R}_3\right) \tag{13}$$

where the perturbing body of mass m_3 is at position $\boldsymbol{R}_3$ relative to the binary centre of mass. This acceleration is substituted in Eqs. (10)–(12) above, using a constant perturber at

$$\boldsymbol{R}_3 = R_3\hat{\boldsymbol{e}}_r \tag{14}$$

where $\hat{\boldsymbol{e}}_r$ is the unit vector towards the third body, together with the standard description of two body motion:

$$\begin{aligned} \boldsymbol{r} &= a\cos E\hat{\boldsymbol{e}}_\xi - ae\hat{\boldsymbol{e}}_\xi + b\sin E\hat{\boldsymbol{e}}_\eta, \\ \dot{\boldsymbol{r}} &= -a\dot{E}\sin E\hat{\boldsymbol{e}}_\xi + b\dot{E}\cos E\hat{\boldsymbol{e}}_\eta, \\ \dot{E} &= \frac{\sqrt{\mu}a^{-3/2}}{1-e\cos E}, \\ dM &= (1-e\cos E)\,dE. \end{aligned} \tag{15}$$

Here E is the eccentric anomaly and M the mean anomaly; a and e are the semi-major axis and eccentricity of the inner binary. Then we have some routine calculations to carry out. Averaging over the complete orbital cycle we can easily see that (Valtonen and Karttunen 2005)

$$\langle \dot{e} \rangle = -\frac{15}{2}\sqrt{\frac{p}{\mu}}\frac{Gm_3}{R_3^3} a\, e(\hat{\boldsymbol{e}}_\xi \cdot \hat{\boldsymbol{e}}_r)(\hat{\boldsymbol{e}}_\eta \cdot \hat{\boldsymbol{e}}_r), \tag{16}$$

$$\langle \dot{p} \rangle = 15\sqrt{\frac{p}{\mu}}\frac{Gm_3}{R_3^3} a^2 e^2(\hat{\boldsymbol{e}}_\xi \cdot \hat{\boldsymbol{e}}_r)(\hat{\boldsymbol{e}}_\eta \cdot \hat{\boldsymbol{e}}_r) \tag{17}$$

which leads to

$$\langle \dot{a} \rangle = \frac{\langle \dot{p} \rangle}{(1-e^2)} + \frac{2pe\,\langle \dot{e} \rangle}{(1-e^2)^2} = 0. \tag{18}$$

Thus, in our current approximation, the semi-major axis of the binary does not have secular evolution, unlike e.g., the eccentricity of the binary.

Our next problem, in order of increasing difficulty, is the calculation of the energy change of a binary when a third body passes by at a close distance to it. It is then not possible to claim that the third body is stationary relative to the binary but we must describe its orbital motion. For that purpose the approximate ways of describing elliptic motion as a function of time are used. It is not as simple as one might expect from the simple geometry of the orbit; one resorts to infinite series, truncating the series at suitable points for practical calculations.

4. Slow encounters

The problem of the energy change of a binary (mass m_B) caused by a passing third body (mass m_3) in rather complicated (Walters 1932a, b, Lyttleton and Yabushita 1965, Yabushita 1966, Heggie 1975, Heggie & Hut 1993, Roy and Haddow 2003). It is obvious that it should be so since the two orbits may be oriented in many different ways relative to each other, with different eccentricities, semi-major axes, closest approach distances etc. For this reason we limit ourselves to the rather simple case of a circular binary and a third body orbit of fixed eccentricity. The outer orbit is taken to be elliptic. The approximation considered here is called *adiabatic* since the perturbing potential varies slowly in comparison with the orbital frequency of the binary.

The inclination between the two orbital planes is denoted ι. The other parameters describing the relative orientations of the inner and the outer orbits Ω and ω, are considered less essential, and their influence is averaged over in the end.

The eccentricity of the outer orbit is denoted e. It has been shown by numerical orbit integration that the energy change is not very sensitive to the eccentricity of the outer orbit near the parabolic case (Saslaw et al. 1974). In the analytical calculation below we give the eccentricity a fixed value $e = 0.265$. This is low enough that the series mentioned above converges well, and the particular value of 0.265 is chosen for convenience since the value of mean anomaly $M = \pi/3$ corresponds to the true anomaly $\phi_e = \pi/2$ at nearly this eccentricity. The calculation could be carried out for any other low value of e without significant change in the final result.

The encounter is assumed to be effective only between $\phi_e = -\pi/2$ and $\phi_e = \pi/2$, which is where most of the action takes place, especially in highly eccentric or parabolic orbits. Therefore we expect that the derived model is applicable to parabolic or even mildly hyperbolic encounters.

The circular binary is rather special but much simpler than the general case because the orientation of the major axis in its orbital plane does not need to be specified. The special symmetry due to the zero inner eccentricity simplifies the derivation considerably.

We divide the integration in two parts: the approaching branch, from $M = -M_0$ to $M = 0$, and the receding branch from $M = 0$ to $M = +M_0$. M_0 is defined so that the inner binary executes exactly one revolution in its initial orbit while the third body progresses from $M = 0$ (pericentre) to $M = M_0$. In our example below M_0 is close to $\pi/3$. We ignore the effects of the subsequent revolutions since they typically happen while the third body is outside the range

$$-\frac{\pi}{2} \leq \phi_e \leq \frac{\pi}{2}. \tag{19}$$

We integrate the relevant functions between $M = 0$ and $M = M_0$ only. Numerical experiments have shown that the effects of the whole encounter may be well estimated by using the receding branch.

After a rather long but straightforward calculation one obtains (Valtonen and Karttunen 2005)

$$\Delta = \frac{\Delta E_B}{E_B} = -\frac{\Delta a_i}{a_i} = -0.18\frac{m_3}{m_B}\left(\frac{Q}{2.5}\right)^{-3}\left[(1+\cos\iota)^2/4\right]\sin 2\phi_0 \tag{20}$$

where a_i is the inner binary semi-major axis, E_B is the binary binding energy, Q is the pericentre distance of the outer orbit, normalised to a_i, and ϕ_0 is the phase angle of the binary at the time of the pericentre passage.

This expression may be compared with results from numerical orbit calculations with parabolic third body orbits. Figure 1 shows an example of how Δ varies with the phase angle ϕ_0. We see that in fact Δ is of the form

$$\Delta = \langle\Delta\rangle + A(\Delta)\sin 2\phi_0, \tag{21}$$

where $A(\Delta)$ is the amplitude of the variation and $\langle\Delta\rangle$ is the mean level.

In addition, the amplitude $A(\Delta)$ should be multiplied by an exponential factor. It becomes

$$\exp[-0.5(Q/Q_1)^{3/2}],$$

where the scale factor Q_1 is

$$Q_1 = 2.5(1 + m_3/m_B)^{1/3}. \tag{22}$$

Therefore the Q dependence in $A(\Delta)$ should be of the form

$$Q^{-3}\exp[-0.5(Q/Q_1)^{3/2}]. \tag{23}$$

Heggie (1975) and Roy and Haddow (2003) derive a coefficient ≈ 3.73 instead of 0.5 in the exponent using a parabolic passing orbit.

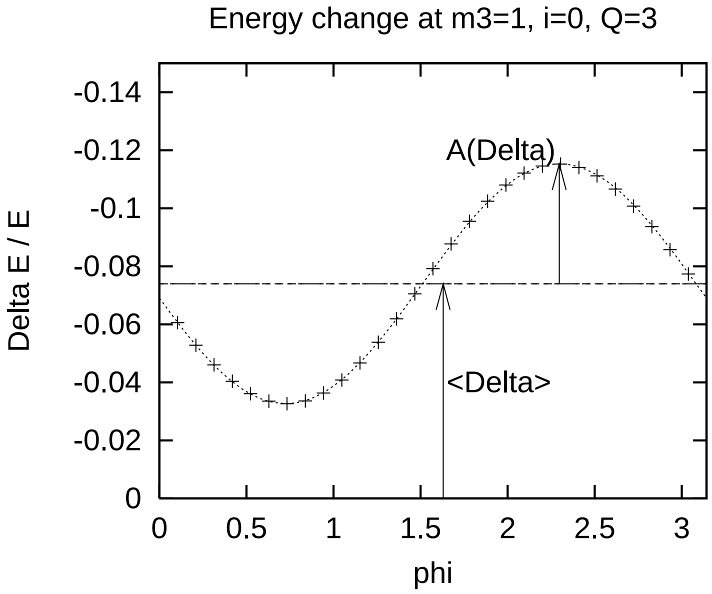

Figure 1. The relative energy change $\Delta E/E = \Delta$ of the binary in a parabolic encounter with a third body. Computer experiments (+) are compared with a sinusoidal function of ϕ_0. The values of the mean energy change $\langle\Delta\rangle$ and the amplitude $A(\Delta)$ are indicated. The case of $\iota = 0^\circ$, $m_3 = m_B$, $e_i = 0$ and $Q = 3$.

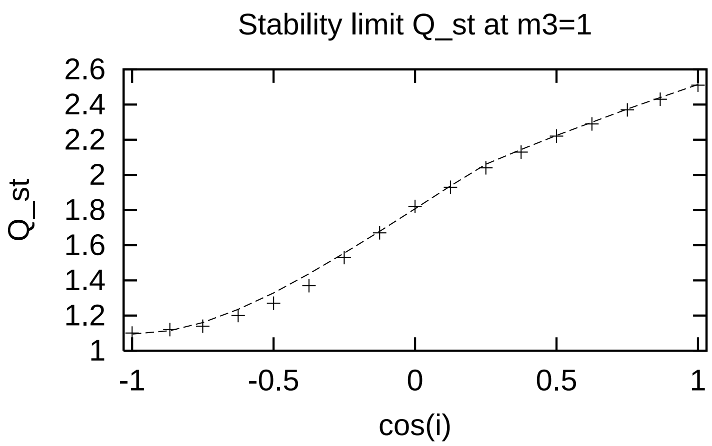

Figure 2. The stability limit in numerical experiments (+) as a function of $\cos\iota$. The line follows Eqs. (25) for the case of $m_3 = m_B$.

In many applications it is easier to use a pure power law rather than this combination of power law and exponential. The exponential can always be modeled locally by a power law, with a power which increases with increasing Q. Thus we have to specify the range of Q we are interested in, and then make the proper choice of the approximate power. We do it first at the smallest values of Q where our theory may still be applicable. This range is approximately

$$Q_{\rm st} < Q < 1.5 Q_{\rm st}$$

where $Q_{\rm st}$ signifies the *stability boundary*. This is defined as the minimum value Q where the original binary survives the encounter at all phase angles ϕ_0 and at all values of ω and Ω.

It is possible to derive an approximate expression for $Q_{\rm st}$ by requiring that the sinusoidally varying component of Δ has a specific value at the stability boundary, e.g. $A(\Delta) = 0.18$. Our argument would be that greater amplitudes of Δ would lead to an exchange of a binary member with the third body. Then

$$Q_{\rm st}^{(a)} = 2.5(m_3/m_B)^{1/3}((1+\cos\iota)^2/4)^{1/3}. \tag{24}$$

Taken literally, this would imply that the stability boundary goes to zero at $\iota = 180^\circ$, which is not reasonable. Actually, in more exact calculations we would expect a functional form: const $+ (1+\cos\iota)^2$ instead of $(1+\cos\iota)^2$. Using this result we write numerical fitting functions

$$\begin{aligned} Q_{\rm st}^{(d)} &= 2.52[(1+m_3/m_B)/2]^{0.45}[(0.1+(1+\cos\iota)^2)/4]^m \\ Q_{\rm st}^{(r)} &= 2.75[(1+m_3/m_B)/2]^{0.225}[(0.4+(1+\cos\iota)^2)/4]^{0.4} \end{aligned} \tag{25}$$

The first form is for direct orbits ($\cos\iota_0 \le \cos\iota \le 1$), and the second one for retrograde orbits ($-1 \le \cos\iota \le \cos\iota_0$). The power law index m is given by

$$m = 0.06 + 0.08(1 + m_3/m_B),$$

and the direct/retrograde border $\cos\iota_0$ is defined as

$$\cos\iota_0 = 1.52[(1+m_3/m_B)/2] - 1.28.$$

This gives a good representation of the stability boundary when the masses are not too unequal, i.e. in the mass range $0.2 \le m_3/m_B \le 2.0$. Note that (m_3/m_B) of $Q_{\rm st}^{(a)}$ is replaced by $(1+m_3/m_B)$ in the more accurate $Q_{\rm st}^{(d)}$ and $Q_{\rm st}^{(r)}$. Because of this the stability boundary scales as a power of the exponential scale factor Q_1 . A fit of these functions to experimental data is shown in Fig. 2.

Hills (1992) determined the stability boundary numerically over the mass range $0.15 \le m_3/m_B \le 5000$. He used orbits of random inclinations and obtained the result

$$Q_{\rm st}(\text{Hills}) = 2.1(1+m_3/m_B)^{1/3}. \tag{26}$$

This is in some ways a compromise between $Q_{\rm st}^{(d)}$ and $Q_{\rm st}^{(r)}$, since the power 1/3 of the mass factor is intermediate between the corresponding powers of 0.45 and 0.225 in our equations.

Notice that our expressions do not contain the mass ratio m_a/m_b of the binary members. Numerical experiments by Hills (1984) for close encounters between a star-planet system and a stellar intruder show that indeed we may neglect this parameter in the first approximation.

5. Stability of triple systems

Up to now the notion of stability has been used in relation to only one pericentre passage. Often it is more interesting to know what happens after many pericentre passages when the third body approaches the binary repeatedly. Obviously, a more stringent stability limit, i.e. a greater value of Q is needed to guarantee stability. The stability may also be defined in different ways, giving slightly different results.

Let us start by defining stability so that for a stable orbit the relative energy change should be no greater than 10^{-3} in either direction during a single pericentre passage. This corresponds to the survival of the triple system for 10^5 revolutions of the outer binary, according to the following estimate. If the destabilising level of accumulated relative energy change is $10^{-1/2}$, and the energy change accumulates in the manner of random walk (Huang & Innanen 1983), then the destabilising level is achieved after $\left(10^{-1/2}/10^{-3}\right)^2 = 10^5$ steps. The random walk type behaviour of the energy changes is due to the phase factor $\sin 2\phi_0$. Generally, successive encounters take place with different values of ϕ_0, the latter being distributed more or less randomly. There is also a constant (independent of ϕ_0) drift factor which may be dominant depending on the inclination. However, numerical experiments (Saslaw et al. 1974) have shown that once the eccentricity e_i has increased to about $e_i = 0.2$, the drift becomes insignificant in comparison with the random walk. The eccentricity goes over this limit quite easily at moderate values of Q.

When we extend the theory to small values of Δ, i.e. to large Q, we have to take account of the exponential factor. It makes the power law Q^{-n} the steeper the greater is the value of Q. For the relative energy change $\Delta = 10^{-3}$ the suitable effective power is 11 (Valtonen 1975).

Therefore the power of the mass and inclination factors in Eq. (24) should be lowered from $1/3$ to ≈ 0.09. Using numerical experiments we further refine the expression and get a new stability limit, suitable for all inclinations:

$$\begin{aligned} Q_{\rm st}(A(\Delta) = 10^{-3}) =& 3.62\,[(1 + m_3/m_B)/2]^{0.23} \left(\frac{m_3}{m_B}\right)^{0.09} \times \\ & [1.035 + \cos\iota]^{0.18}\,. \end{aligned} \tag{27}$$

Figure 3 shows that this gives a good description of the stability boundary. Figure 3 also demonstrates that the stability limit based on the drift $|\langle\Delta\rangle|$ at the level of 10^{-3} is about equal to or less than the limit derived above from the amplitude $A(\Delta)$. Thus the contribution of the drift to the stability boundary can generally be ignored. Only at small inclinations and close to $\iota = 180°$, and as long as e_i stays small, does the drift become important.

Less stringent stability criteria have also been used. One may require that in 100 revolutions of the outer orbit there is no major orbital change (Mardling & Aarseth 1999), or that within some specific number of revolutions of the original outer orbit there should be neither exchanges of the binary members nor escapes of any of the bodies (Huang & Innanen (1983) use the revolution number $N = 62$, Eggleton and Kiseleva (1995) use $N = 100$); sometimes the survival through an even smaller number (10–20) of revolutions has been considered to

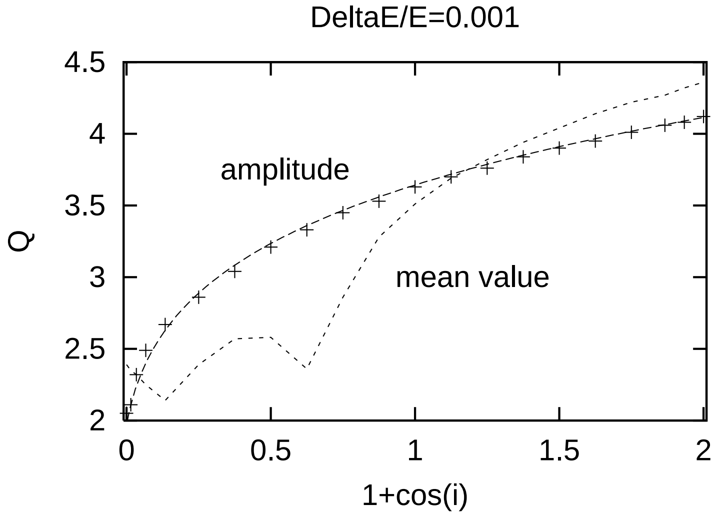

Figure 3. The Q-boundary of $A(\Delta) = 10^{-3}$ (+) and of $|\langle\Delta\rangle| = 10^{-3}$ (dotted line). The analytical function for the former is drawn as a dashed line. The mean value changes its sign from positive to negative at $\cos\iota \approx -0.375$ when going from -1 to $+1$ along the $\cos\iota$ axis. The case of $m_3 = m_B$.

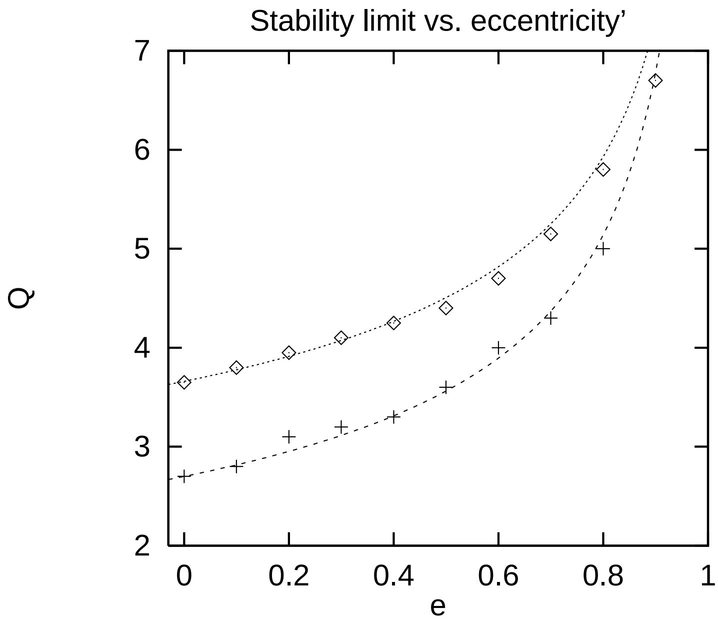

Figure 4. A comparison of Eq. (28) with experimental data from Huang & Innanen 1983 (+) and Mardling & Aarseth 1999 ($\diamond$).

be sufficient for stability (Harrington 1972, 1975). However, the results are not very different even though the chosen revolution number N varies a lot. This is because in a random walk with a constant energy step, in order to cover a standard magnitude change in energy, $\sqrt{N} = \mathrm{const}\, Q^{11}$, i.e. the stability limit Q_{st} varies only as the 1/22nd power of N.

The stability limit also depends on the strength of binding of the outer binary to the inner binary. If the outer binary is initially only very loosely bound, then even a small positive energy increase at the pericentre may set it loose. The degree of relative binding is best described by the axial ratio a_i/a, or since $Q = (a/a_i)(1-e)$, by $(1-e)/Q$. Putting this relative binding equal to the relative energy change $|\Delta| \propto Q^{-11}$, i.e. setting $|\Delta| = \Delta E_B/E_B \approx E_{\mathrm{outer}}/E_B \approx (1-e)/Q$, we find that the stability limit varies as

$$Q_{st} \propto (1-e)^{-\alpha} \tag{28}$$

where $\alpha = 0.1$. Actually, putting $\alpha = 0.3 - 0.4$ gives a better agreement with some experiments (see Fig. 4, Huang & Innanen 1983, Mardling & Aarseth 1999) while in others $\alpha \simeq 0.0$ (Eggleton & Kiseleva 1995); the results depend on the definition of stability and on the masses of the bodies.

The experimental value for the stability limit for equal masses $m_1 = m_2 = m_3$, $e = 0$ and $\cos\iota = 1$ is $Q = 2.7$ after $N = 62$ revolutions (Huang & Innanen 1983). With these parameters $A(\Delta)$ gives the value 0.03 which becomes $\Delta E_B/E_B = 0.23$ after multiplication by $\sqrt{62}$. This is just at the relative energy change usually associated with instability (i.e. ≈ 0.2) and thus the stability limit of Huang & Innanen (1983) is as expected. The stability limit of Mardling & Aarseth (1999) for the same case is $Q = 3.65$ which seems contradictory. At this pericentre distance we expect the average energy change per revolution to be 0.003 which is multiplied by $\sqrt{100}$ and adds up to 0.03 after 100 revolutions. This is only about 10% of the value at the stability boundary. However, in Mardling & Aarseth (1999) the stability criterion was such that two orbits initially differing by one part in 10^5 in the eccentricity should remain close after 100 orbits. $|\Delta E_B/E_B|$ being at about 10% of the stability limit could well be used as a definition for two nearly identical orbits not to have evolved too far apart from each other and for the system to be stable. The corresponding stability limit of Bailyn (1984), $Q = 3.1$ lies between the previous two, and it appears that the definition of instability is also intermediate between Huang & Innanen (1983) and Mardling & Aarseth (1999).

So far we have not considered the possibility that the inner binary orbit may be eccentric. In the first approximation we may take the time averaged mean separation $r = a_i[1 + 0.5e_i^2]$ in place of a_i in our perturbation equations. Then the stability limits obtained earlier are simply multiplied by r/a_i since the inner binary is effectively this much greater in extent, and the encounter has to be more distant by the same factor for stability. In practice it appears that this method gives reasonable agreement with numerical experiments (Bailyn 1984).

Even though there are obviously many different ways to define stability, and correspondingly many possible stability limits, it appears safest to use Q_{st} as defined by orbit calculations:

$$Q_{\rm st}^{(2)} = 3\left(1+\frac{m_3}{m_B}\right)^{1/3}(1-e)^{-1/6} \times\left(1+\frac{1}{2}e_i^2\right)\left[2-(\cos\iota)^2\right]^{1/3}. \tag{29}$$

This expression has been found to be quite satisfactory in numerical experiments (Valtonen et al. 2006) for the stability of $N = 10000$ revolutions when the stability is defined in the manner of Huang and Innanen (1983). The expression is valid in the range $1/6 \le m_3/m_B \le 5$ and $0 \le e \le 0.9$.

Here we must remember the Kozai resonance which operates effectively at inclination angles close to $\iota = \pi/2$. The inner eccentricity grows up to values close to $e_i = 1$ which means that the factor $1 + 0.5e_i^2$ is best replaced by 1.5 in Eq. (29) for these inclinations, independent of the original eccentricity e_i (Miller and Hamilton 2002, Wen 2003).

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