Few-Body Problem: Theory and Computer Simulations
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From the 3-Body Problem to a Three-Hundred-Billion-Body Universe: A Perspective

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Abstract. This is a brief sketch of five major fundamental insights into the cosmological many-body problem. It starts with the three-body problem on small scales, then progresses through the few-body problem to the largest observed clusters of galaxies and clustering on the largest scales. The five insights involve: 1. The discrete nature of the gravitational field, 2. The slingshot instability of the 3-body problem, 3. Connections between correlation functions and dynamics, 4. Distribution functions and their derivation from gravitational thermodynamics, and 5. A statistical mechanical basis for the gravitational thermodynamics.

Starting with the dynamics of a few particles, we can build up a series of insights into the behaviour of the cosmological many-body problem, and its relation to galaxy clustering. Here I'll give a somewhat impressionistic summary of these developments during the last two decades. They are all based on a first discussion of this problem by Isaac Newton replying to a letter from Richard Bentley in 1692. Bentley, one of England's leading theologians and classical scholars (who applied the long disused digamma letter of the Greek alphabet to correct and reinterpret earlier texts of Homer and other classics), had asked Newton whether a universe filled uniformly with gravitating particles would be stable.

Newton's replies (cf Saslaw 2000 for more technical details about this and many of the topics mentioned here) clearly showed his appreciation of the difference between finite and infinite gravitating systems. Both were unstable. In a finite universe the particles (stars to Newton, galaxies to us) would eventually form a great spherical collection at the centre. But in an infinite universe, they would collect into many clusters scattered throughout that whole universe. Our more modern expression of this is that finite spherical systems are rotationally invariant only around a single point, their centre, and not translationally invariant anywhere. Infinite systems, however, are translationally and relationally invariant everywhere, provided the scale is large enough to be statistically homogeneous (i.e. that samples large enough to be representative have essentially the same statistical properties anywhere in the system).

Thus we would expect properties like correlation functions, spatial and velocity distribution functions, mass (or luminosity) segregation, and the geography of dark matter to behave quite differently in finite and infinite systems. Although Newton recognized this qualitatively, it has taken nearly three centuries to build a quantitative understanding, mostly developed in the last half century.

Starting with the simplest version of Bentley's cosmological many-body problem, a Poisson distribution of particles (galaxies) in an infinite expanding Einstein-Friedman universe, consider how this distribution evolves. The first insight is that the zeroth-order approximation to the gravitational field is not smooth, but grainy. A grainy system has many more types of instabilities than a continuous system. On small scales, these instabilities can produce non-linear chaos and emergent structures on the local dynamical timescale $(G\rho)^{-1/2}$ where ρ is the average local density. In almost all initial discrete distributions (a lattice being an exception), neighbouring particles will be closer than the global average in some regions and more diffuse elsewhere.

In regions where two particles are closer than average, they can remain gravitationally bound, especially if they are in virial equilibrium so that their kinetic and potential energies satisfy $2K + W = 0$. Then the cosmic energy equation, $d(K + W)/dt + (2K + W)\dot{R}/R = 0$ where $R(t)$ is the cosmic scale length, shows that the total energy of an isolated virialized cluster does not change as the universe expands. The cosmic energy equation is a moment of the exact equations of motion for particles in the expanding universe, and is very general, applying to nonlinear systems with $\delta\rho/\rho \gg 1$ as well as to linear ones. It also shows how a group's total energy is affected by the adiabatic expansion of the universe if the group is not virialized, and how these dynamics are affected by the Hubble parameter $\dot{R}/R$ of the background universe. The Hubble parameter depends on the total density of the universe. This includes its dark matter and energy. If these are uniformly distributed, they will influence local clustering mainly through $R(t)$, but if these are inhomogeneous or time dependent, they will complicate clustering considerably. Although bound binaries can form by chance, most easily in flat universes having zero total energy, they are more generally the result of unstable triplets.

The three-body problem promotes our second insight into early clustering. Its dynamical slingshot instability breaks the triplet into a more tightly bound binary and an escaping particle, usually after a few to a few hundred orbital periods, depending on the initial orbit parameters. Binaries then interact with other binaries, single particles, triplets, etc. to produce a variety of larger and larger groups and clusters in more extended regions over longer timescales. The system is unstable to clustering, and the clusters themselves are unstable to ejecting their higher energy particles, but over increasingly long timescales. This is a form of dynamical dissipation whose increasing entropy leads to complex chaotic non-linear motions, and the system becomes inhomogeneous over increasing scales. Underdense regions form between clusters.

Replication of these dynamics using computer simulations which directly integrate the mutual gravitational orbits of all the particles in comoving coordinates, helps to understand the development of galaxy clustering. Digital simulations on Earth, the analog simulation in the sky, and fundamental physical theory can all be compared to keep our insights from going too far astray. The basic problem is to find the best types of information to compare quantitatively. Pictorial illustrations, either in space or on the sky, are subject to eye-brain illusions and can be misleading. Overly detailed information, such as the positions and velocities of all individual particles, is unique to a particular realization and does not provide much generic insight. The art in understanding many-body problems consists of finding descriptions which can be related to fundamental physical concepts and can be calculated relatively easily.

Of the many statistical descriptions proposed to quantify Newton's cosmological many-body problem, two have been directly related to fundamental gravitational physics. In the 1950's, the two-particle correlation function, $\xi(r)$, was adapted from theories of imperfect gases and turbulence as a description of galaxy clustering. It was first measured accurately by Totsuji and Kihara (1969). They realized from thermodynamic theory that the universe resembled a system in neutral equilibrium where cosmic expansion nearly balances the gravitational attraction of linear density enhancements. Under these conditions, the two-particle correlation function was already known to have the scale-free form of a simple power law. Many subsequent analyses have confirmed and extended their work which gave $\xi(r) = (r/r_o)^{-1.8}$ with $r_o = 4.7$Mpc. Correlation functions provide a third major insight into galaxy clustering.

Like other statistics, the low order correlation functions can be observed in the sky and measured in simulations. But they can also be partly derived from the $6N$ comoving equations of motion of N particles in the position-momentum phase space of an expanding universe. These orbital equations are also compactly represented by Liouville's equation. Projecting Liouville's $6N$ dimensional equation into lower $6m$-dimensional phase spaces by integrating over $6(N - m)$ coordinates, gives a set of coupled non-linear partial differentio-integral equations known as the BBGKY hierarchy. These can be solved, with suitable approximations, for the low-order correlation functions in the linear regime where their amplitudes are less than about unity. With additional approximations, some solutions can be obtained in the non-linear regime which characterize the observations at low redshifts on scales $\lesssim$ 10Mpc. Details of these solutions depend on the initial distribution of points and the rate of expansion of the universe. They may also be modified by non-uniform dark matter with a different distribution than the galaxies (e.g. in massive haloes containing many galaxies).

Analytic solutions of the BBGKY hierarchy are very complicated, and low-order correlation functions contain very restricted information. After all, a cluster of 100 particles involves the 100-point correlation function, as does an underdense region which is expected to contain 100 particles but doesn't. These problems led to a search for more informative statistics which were easier to calculate, especially in the non-linear regime.

Actually such a statistic had been used implicitly by Herschel (1784) and explicitly by Hubble (1936) who simply counted the numbers of galaxies in cells of a given size and shape on the sky. These counts-in-cells are one avatar of the spatial distribution function $f(N)$, which is the probability that there are N particles in a cell. It can be applied to cells in space, and cells on the sky (which are conical projections of spatial cells); it also represents the continuous probability for finding a particle at a given distance from an arbitrary point in space or on the sky. It is related to volume integrals of correlation functions of all orders. Although distribution functions had long been known and are easy to measure, they fell into disuse because they had not been related to the underlying dynamics.

This has now changed, first through the development (Saslaw & Hamilton 1984; Saslaw 2000) of a thermodynamic derivation of $f(N)$ for the cosmological many-body problem, and second through its more recent connection to statistical mechanics (Ahmad, Saslaw & Bhat 2002; Leong & Saslaw 2004). These provide a fundamental physical understanding which turns out to be especially interesting because of its observed agreement with galaxy clustering (e.g. Sivakoff & Saslaw 2005). Thermodynamics provides our fourth major insight.

The thermodynamic and statistical mechanical theories both have a simple general physical assumption in common. This is that the local dynamical timescale $(G\rho)^{-1/2}$ in an overdense region is faster than the global gravitational timescale $(G\bar{\rho})^{-1/2}$, for the average density $\bar{\rho}$, and this difference makes it possible for gravitational clustering to evolve through a sequence of quasi-equilibrium states. In other words, local equilibria can arise faster than the cosmic expansion can disrupt them. The difference in timescales does not have to be large for this to happen, as direct N-body simulations show. Most of the general relativity cosmologies, including those with a cosmological constant and quintessence, satisfy this criterion. From a statistical mechanical point of view, this means that the system can undergo rapid (compared with the Hubble timescale) microscopic transitions which sample all its accessible states with approximately equal a priori probability.

Consequently, average macroscopic thermodynamic and statistical quantities, such as temperature, pressure, density, chemical potential, internal energy, etc., can be reasonably defined along with local fluctuations around these averages. At any given time, there will be thermodynamic equilibrium relations such as equations of state among these quantities. Even though the macroscopic quantities may themselves change over the longer timescale, they continue to satisfy these equilibrium relations to a good approximation at any given time. This is the essence of quasi-equilibrium evolution. It is analogous to the case of ordinary non-equilibrium thermodynamics. The main conditions which could prevent quasi-equilibrium evolution are initial non-equilibrium inhomogeneities on scales over which $f(N)$ is being measured.

When quasi-equilibrium holds we can derive the equations of state, including gravitational interactions of the point masses, and apply standard thermodynamic fluctuation theory to them. The constraint that the system satisfies the first and second laws of thermodynamics gives a unique form for the distribution function $f(N)$, which describes quasi-equilibrium fluctuations around the average value of $\bar{N} = \bar{n}V$:

$$f(N) = \frac{\bar{N}(1-b)}{N!}[\bar{N}(1-b) + Nb]^{N-1} e^{-[\bar{N}(1-b)+Nb]} \tag{1}$$

where

$$b = \frac{-W}{2K} = \frac{2\pi G m^2 n}{3T} \int_0^R \xi(r) r dr \tag{2}$$

and $0 \le b \le 1$. Here b is the ratio of the negative of the gravitational potential correlation energy, $-W$, to twice the kinetic energy. This is written explicitly here for spherical volumes of radius R with kinetic temperature T on the right hand side, as an example. Note that when the gravitational potential is absent, $b = 0$ and $f(N)$ reduces to a Poisson distribution. So b is a measure of clustering, and it can also be determined self-consistently from the variance of counts-in-cells for Eq.(1). There is also a self-consistent peculiar velocity distribution function for Eq.(1); this is a generalization of the Maxwell-Boltzmann distribution to the cosmological many-body problem. Many properties of these distributions are summarized in Saslaw (2000). They agree very well with both direct N-body computer simulations, and with observations. Recent observational determination of b at redshifts less than about 0.1, using about 650,000 galaxies from the 2MASS survey, gives $b = 0.867 \pm 0.026$ from counts-in-cells with sides of 8 degrees on the sky (Sivakoff & Saslaw 2005). This scale encompasses most of the correlation contribution to b in Eq.(2). Equation (1) applies in linear and non-linear regimes to cosmological N-body dynamical systems which cluster from a wide range of initial conditions. However a mathematical description of the basin of attraction for Eq.(1) remains a fundamental question for the theory.

Although thermodynamics provides a fourth major insight into the cosmological many-body problem, we can go still deeper. At a more microscopic level, statistical mechanisms determine and generalize the thermodynamics, providing a fifth major insight. To derive a statistical mechanics, we need to solve the partition function. This was long thought to be impossible for gravitating systems in general, and for the cosmological case in particular. There were two reasons for this belief. The first was that at very short distances the $1/r$ point mass gravitational potential becomes infinite and makes the partition function diverge. The cure for this is to soften the potential to the form $(r^2 + \epsilon^2)^{-1/2}$ so that it becomes constant at small r. This might seem artificial, but in fact it corresponds to surrounding the particle with a halo which, in the case of galaxies, is a simple approximately isothermal sphere of dark matter for which there is independent astronomical evidence. In any case, it is possible to calculate the partition function analytically with this potential (Ahmad, Saslaw & Bhat 2002). Then one can let $\epsilon \to 0$ and show that the results remain finite and converge uniformly in this limit to the earlier thermodynamic results. As a bonus one sees the effects of haloes for $\epsilon > 0$. Moreover, the partition function provides a clear approximation scheme for the effects of different levels of many-body clusters on the statistical mechanics (Ahmad, Saslaw & Malik 2006).

The second reason people doubted the existence of a gravitational cosmological many-body partition function is that in an effectively infinite homogeneous system the number of particles in a shell increases as $r^2 dr$ while the potential decreases only as r^{-1}. So the integral of the total potential over an infinite volume, which occurs in the partition function, diverges. What this view forgets, however, is that this divergence, which represents the mean field, is exactly cancelled by the expansion in a large class of universes (see Saslaw & Fang 1996; Saslaw 2000). This occurs because the cosmic expansion may be represented, in these models, as resulting from a potential which has the same amplitude, but opposite sign, as the smoothed gravitational background. This holds for all values of the cosmic curvature, and for models which may contain a cosmological constant or gravitating quintessence. It implies that local pecular velocities in proper coordinates depend only on local fluctuations of the gravitational force. As long as these fluctuations are not correlated on infinite scales, the partition function will converge. It may depend on the size and shape of a volume, especially if it has a dimension less than the correlation length.

All the thermodynamic functions, as well as quantities such as $f(N)$ can be derived analytically from the partition function, although they may be singular at $b = 1$ which represents a first order phase transition (Baumann, Leong & Saslaw 2003). In the limit when the small scale softening ϵ of the potential vanishes, these reduce to the results derived from the thermodynamic arguments alone, confirming and generalizing them. Incorporating haloes of galaxies modifies $f(N)$ and the velocity distribution. Comparing these modifications with observations (Leong & Saslaw 2004) shows that the observations are much more readily consistent with most of the dark matter surrounding individual galaxies as they cluster, rather than with many galaxies forming or remaining within a single supermassive dark matter halo.

The statistical mechanical partition function also provides a rigorous approximation procedure, within quasi-equilibrium conditions, for deriving the thermodynamic results. Two main approximations were employed. The first was to neglect the irreducible triplet and higher order terms in a cluster expansion of the partition function. Recently it has become possible to calculate these terms explicitly and show that they are usually small, becoming negligible for systems with large numbers of particles (Ahmad, Saslaw & Malik 2006). The second approximation was to neglect higher order terms in the expansion of the interaction exponential of the Hamiltonian; these too are now found to be negligible for systems with large N. Thus the earlier thermodynamic approximations are now known to be both physically and mathematically reasonable, and their small correction terms are known.

So over the last two decades we have understood much about the physics of the cosmological many-body problem and its relation to galaxy clustering. It started by relating the low order correlation functions to the particle dynamics through the BBGKY hierarchy. Then, when that analysis became too difficult, particularly for higher order non-linear correlations, we switched to a more macroscopic thermodynamic approach which gives many further insights. More recently, it has been possible to develop a statistical mechanical approach, intermediate between the detailed microscopic level dynamics and the thermodynamic description.

Encouraged by their consistency with observations, these approaches can all be utilized to help understand other fundamental related questions. An example would be a more direct relation of the cosmological statistical mechanics to its underlying gravitational dynamics. How does quasi-equilibrium constrain the sum of energy states in the partition function? This, in turn is closely connected to determining the basin of attraction for the distribution function of Eq.(1). The basin of attraction provides insight into the set of initial conditions which would evolve, through dynamical dissipation, into the observed distribution of the galaxies. Cosmological models, and galaxy formation within them, would then have to be consistent with such initial conditions.

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