Few-Body Problem: Theory and Computer Simulations
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Gravitational Structure Formation

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Abstract. We discuss the formation of the first structures in gravitational N-body simulations. The role of two-body interaction is found to be a crucial element and an analogy with the dynamics of the Coulomb lattice, well-studied in solid state physics, is discussed.

The standard models of the formation of large scale structure of the universe are based on the gravitational growth of small initial density fluctuations in a homogeneous and isotropic medium (e.g. Peebles 1980). For example, in the so-called Cold Dark Matter (CDM) model, particles interact only gravitationally and are 'cold', i.e. with very small initial velocity dispersion. This situation allows one to model this system with a collisionless Boltzmann equation and, for sufficiently large scales, pressure-less fluid equations. These fluid equations can be solved in a perturbative way for small density fluctuations (see e.g. Peebles 1980). However, this treatment is inapplicable in the strong non-linear regime. Then, the most widely used tool to study gravitational clustering in the various regimes is N-body simulations (NBS) which are based on the computation of the dynamics of self-gravitating particles in expanding universe.

These simulations can be performed by considering an infinite periodic system, i.e. a finite system with periodic boundary condition. Despite the simplicity of the system, in which dynamics are Newtonian on all but the smallest scales, the analytic understanding of this crucial problem is limited to the regime of very small fluctuations where a linear analysis can be performed. In the cosmological case, the problem can be approximated to Newtonian but the equation of motions are modified because of the expanding background (Peebles, 1980). As discussed below, we find it instructive to consider some simplified cases where the expansion is not included and then study the differences introduced by space expansion.

An important point should be stressed: for numerical reasons due to computer limitations, the cosmological density field must be discretized into "macro-particles", interacting gravitationally, and which are tens of orders of magnitude heavier than the (elementary) CDM particles. This procedure introduces discreteness at a much larger scale than the discreteness inherent to the CDM particles. By discreteness we mean statistical and dynamical effects which are not described by the self-gravitating fluid approximation. The discreteness has different manifestations in the evolution of the system (see e.g. Baertschiger et al., 2002 and references therein). In this context it is necessary to consider the issue of the physical role of discrete fluctuations in the dynamics, which go beyond a description where particles play the role of collisionless fluid elements and the evolution can be understood in terms of a self-gravitating fluid.

In order to study the gravitational many-body problem, we have considered a paradigmatic system consisting of a very simple initial particle distribution represented by a slightly perturbed, simple cubic lattice with periodic boundary conditions and zero initial velocities (Joyce et al., 2005). A perfect cubic lattice is an unstable equilibrium configuration for gravitational dynamics, as the force on each particle vanishes. A slightly perturbed lattice (see Fig.1) represents instead a situation where the force on a particle is small and linearly proportional to the displacements of all the particles from their lattice positions. When the system is evolved for long enough times, complex non-linear structures arise, as shown in Fig.1 (Right Panel). While the full understanding of this clustering dynamics is not currently available, some steps have been made for what concerns the evolution of the system at early times (Joyce et al., 2005, Baertschiger & Sylos Labini, 2004).

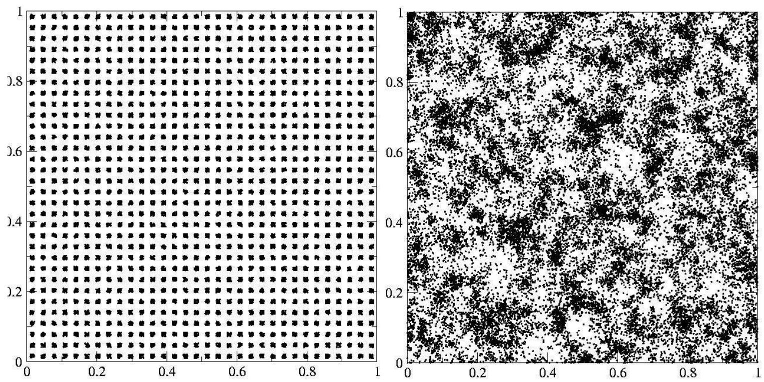

Figure 1. *Left Panel*: Slightly perturbed lattice with zero initial velocity dispersion. This is an orthogonal projection of a system with 32^3 particles. The force on a particle is small and linearly proportional to the displacements of all the particles from their lattice positions. *Right Panel*: When the system is evolved under its own gravity for long times it creates complex non-linear structures characterized by the presence of clusters of different sizes. When the size of the largest cluster becomes of the order of the box size, the simulation is dominated by finite size effects.

The characterization of the gravitational evolution of this system for small displacements, i.e. up to when two nearest particles collide, can be achieved by a perturbative theory (Joyce et al., 2005). Up to a change in sign in the force, the initial configuration is identical to the Coulomb lattice (or Wigner crystal) in solid-state physics (see e.g. Pines, 1963): by using standard techniques of solid-state physics it is possible to develop an approximation to the evolution of the gravitational many-body problem. The equation of motion of particles moving under their mutual self-gravity in a static universe is

$$\ddot{\mathbf{x}}_i = -\sum_{i \neq j} \frac{Gm_j(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3}\,, \tag{1}$$

where for the sake of clarity we have not written explicitly the sum on the replicas of the system. Here dots denote derivatives with respect to time t, $\mathbf{x}_i$ is the position of the ith particle, of mass m_i [1]. Perturbations from the Coulomb lattice are described simply by Eq. (1), with $Gm^2 \to -e^2$ (where e is the electronic charge). By denoting $\mathbf{x}_i(t) = \mathbf{R} + \mathbf{u}(\mathbf{R}, t)$ where $\mathbf{R}$ is the lattice vector of the ith particle and $\mathbf{u}(\mathbf{R}, t)$ is the displacement of the particle from $\mathbf{R}$, and by expanding to linear order in $\mathbf{u}(\mathbf{R}, t)$ about the equilibrium lattice configuration (in which the force on each particle is exactly zero), we obtain

$$\ddot{\mathbf{u}}(\mathbf{R}, t) = -\sum_{\mathbf{R}'} \mathcal{D}(\mathbf{R} - \mathbf{R}')\mathbf{u}(\mathbf{R}', t)\,. \tag{2}$$

In solid state physics, the matrix $\mathcal{D}$ is called, for any interaction, the *dynamical matrix* which, according to the Bloch theorem, is diagonalized by plane waves in reciprocal space. The spectrum of eigenvalues is complex and as in the case of the Coulomb lattice, eigenvectors show the characteristic branch structure.

For example, in the Coulomb lattice there is the *optical* branch, describing oscillations with plasma frequency $\omega_p^2 = 4\pi e^2 n_0/m$ (where n_0 is the electronic number density) which in the gravitational case corresponds, for long wavelength perturbations, to the evolution of instabilities predicted by an analogous fluid description of the self-gravitating system (Joyce et al., 2005). Further it is possible to characterize precisely, up to when two nearest particles collide, the deviations from this fluid-like behaviour at shorter wavelengths arising from the discrete nature of the system. For instance, there are also oscillating modes, and modes which grow faster than the fluid one, which are absent in the fluid description.

This analysis should be a useful step toward a precise quantitative understanding, which is currently lacking, of the role of discreteness in cosmological NBS (see e.g. Melott and Shandarin, 1993). These simulations are most usually started from configurations which are simple cubic lattices perturbed in a manner prescribed by a theoretical cosmological model and thus, at early times, dynamical evolution can be studied as the paradigmatic case discussed above,

[1] Note that as written in Eq. (1) the infinite sum giving the force on a particle is not explicitly well defined. It is calculated by solving the Poisson equation for the potential, with the mean mass density subtracted in the source term. In the cosmological case this is appropriate, as the effect of the mean density is absorbed in the Hubble expansion; in the case of the Coulomb lattice it corresponds to the assumed presence of an oppositely charged neutralizing background. In the non-expanding case the negative background can be intended as a trick used to make the potential finite: however in the conditions we consider this does not affect the computation of the force.

with only a simple modification of the dynamical equations due to the expansion of the Universe. The main difference is quantitative, namely in the expanding case the growth of perturbations is power-law in time while in the non-expanding case it is exponential (Joyce et al., 2005). Apart from this, no qualitative physical difference in the formed non-linear structures is apparent (see also Sylos Labini et al., 2004).

One of the central questions in the context of gravitational NBS is whether one may have some analytical predictions which relate the observed power-law in the correlation function of the particles at late times with some features of the initial particle configuration. For example it has been recently observed (Sylos Labini et al., 2004) that in a broad class of gravitational NBS a universal behaviour in the non-linear clustering develops, characterized by the *exponent* of the conditional density. This statistical quantity is defined as

$$\langle n(r) \rangle_p = \frac{\langle n(r) n(0) \rangle}{\langle n(0) \rangle} \ , \tag{3}$$

so that $\langle n(\vec{r}) \rangle_p \, dV$ gives the a-priori probability of finding 1 particles placed in the infinitesimal volumes dV around $\vec{r}$ with the *condition* that the origin of coordinates is *occupied* by a particle, i.e. it represents the average density of particles seen by a fixed particle at a distance r from it. Once power-law correlations are developed, i.e. $\langle n(r) \rangle_p \approx r^{-\gamma}$ with $\gamma \approx 1.8$, the subsequent evolution increases the range of scales where non-linear clustering is formed, i.e. where $\langle n(r) \rangle_p \gg n_0$, by approximatively a simple rescaling: denoting by $\langle n(r,t) \rangle_p$ the conditional density at time t, one has

$$\begin{aligned} \langle n(r, t+\delta) \rangle_p &\approx \langle n(a \cdot r, t) \rangle_p \\ \langle n(r) \rangle_p &\approx r^{-\gamma} \ \text{ for } \ r < \lambda_0(t) \end{aligned} \tag{4}$$

where $a < 0$ is a constant which depends on the time (Baertschiger and Sylos Labini, 2004) and $\lambda_0(t)$ is the crossover scale between strong ($\langle n(r) \rangle_p \gg n_0$) and weak ($\langle n(r) \rangle_p \approx n_0$) clustering. While the constant a, as well as $\lambda_0(t)$, depends on the particular system considered it seems that the exponent γ is the same in many different cases (Sylos Labini et al., 2004). Thus an important element of the nature of clustering in the non-linear regime can be associated with what is common to all these different simulations: their evolution in the non-linear regime is dominated by fluctuations at small scales, which are similar in all cases at the time this clustering develops. Such "shot-noise" fluctuations are in fact intrinsic to any particle distribution. This corresponds to domination by nearest neighbour interactions when the first non-linear structures are formed (Baertschiger and Sylos Labini, 2004).

To study in detail this early non-linear dynamics, i.e. the growth of the first non-linear correlations, we considered the gravitational evolution of a cold particle distribution with no correlations, i.e. a Poisson configuration (Baertschiger and Sylos Labini, 2004). One may show that by treating this simple case in a static universe as an ensemble of isolated two-body systems, one may understand the origin of the first non-linear correlated structures. This is possible because: (i) the full gravitational force probability distribution approaches the nearest-neighbour force probability distribution at large values of the field and (ii) most of particles are mutually nearest-neighbours. The exponent of the conditional density is then simply related to the functional form of the time evolved nearest neighbours probability distribution, whose time dependence can be computed by using Liouville theorem for the gravitational two-body system (Baertschiger and Sylos Labini, 2004).

The fundamental open problem is that of understanding whether large non-linear structures, which at late times contain many particles, are produced solely by collisionless fluid dynamics, or whether the particle collisional processes (i.e. discreteness effects) are important also in the long-term, or whether they are made by a mix of these two effects. Important elements in this respect are represented by the fact that the correlation function of the evolved system has a strikingly similar functional form to the one generated at early times by two-body interaction (described by Eq.4) and by the fact that aggregation proceeds in a hierarchical, bottom-up manner (Baertschiger and Sylos Labini, 2004).

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