*Few-Body Problem: Theory and Computer Simulations*
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# Measuring the Rényi Information and Multifractality of the Galaxy Distribution observed by Spitzer Space Telescope

Fan Fang

*Spitzer Science Center, California Institute of Technology*

**Abstract.** We review the concepts of Rényi information and multifractals, and discuss their relation to the probability distribution function. These measurements can provide a complete statistical description by probing the higher moments of the galaxy distribution. We apply these high-moment measurements to mid-infrared samples of galaxies detected by the Infrared Array Camera onboard the Spitzer Space Telescope, and discuss in brief the implications for computer simulations.

## 1. Probability Distribution, Rényi Information, and Multifractals

The galaxy spatial distribution, when seen as an N-body process in an expanding universe, is most often described statistically and subsequently physical interpretations sought (Saslaw, this volume). The probability distribution function, describing the probability of finding a given number of galaxies in a given-size volume, is the most general description which contains all the moments of the galaxy spatial distribution. These statistical moments can be polymorphically measured or described by other statistical properties, such as the Rényi information and multifractals.

Suppose we cover a sample of $N_g$ galaxies by a collection of $N_c$ non-overlapping cells. The $\beta$-order Rényi information $I_\beta$ is defined as (Rényi 1970)

$$I_\beta = \frac{1}{\beta - 1} \log \Sigma p^\beta, \tag{1}$$

where $p$ is the probability of finding a galaxy in a cell, and the sum is over all cells. The definition shows the relation between the $\beta$-order Rényi information and the $\beta$-moment. More specifically, the $\beta$-order Rényi information can be written as

$$I_\beta = \frac{1}{\beta - 1} (\log \frac{N_c}{N_g^\beta} + \log \Sigma_i N_i^\beta f(N_i)), \tag{2}$$

where $f(N)$ is the probability distribution function, and the sum is now over numbers of galaxies contained in the cells.

Mathematically the multifractal dimensions can be defined as

$$D(\beta) = \lim_{r \longrightarrow 0} \frac{I_\beta(r)}{\log r}, \tag{3}$$

where $r$ is the cell size. Physically the limit cannot be achieved as $r$ is bounded by the average galaxy separation. Practically the multifractal dimensions are measured by the slopes of the $I_\beta(r)$ versus $\log r$ relation, which can be established by experiment. Note that this definition is consistent with the more familiar one based on the generalized correlation integral. The multifractals, apart from their geometrical image of spatial occupation at multiple orders, have a clear statistical meaning as measures of the moments of a probability distribution.

## 2. Results from the Spitzer First Look Survey

A previous study (Fang et al 2004) measured the 2-point angular correlation functions for galaxy samples of all four wavelengths of the Infrared Array Camera (IRAC) in the Spitzer First Look Survey (FLS). Measuring the second moment, 2-point statistics does not completely describe a non-Gaussian spatial density field. Here we extend the study to measuring high moments for the same samples, using the probability distribution function, the Rényi information, and the multifractal descriptions. For all measurements the two-dimensional area covered by the IRAC samples is divided into contiguous square cells of varying sizes. We discard cells that contain invalid mosaic pixels, from the masks that were used to establish the IRAC samples. The boundary effect and selection bias are at minimum.

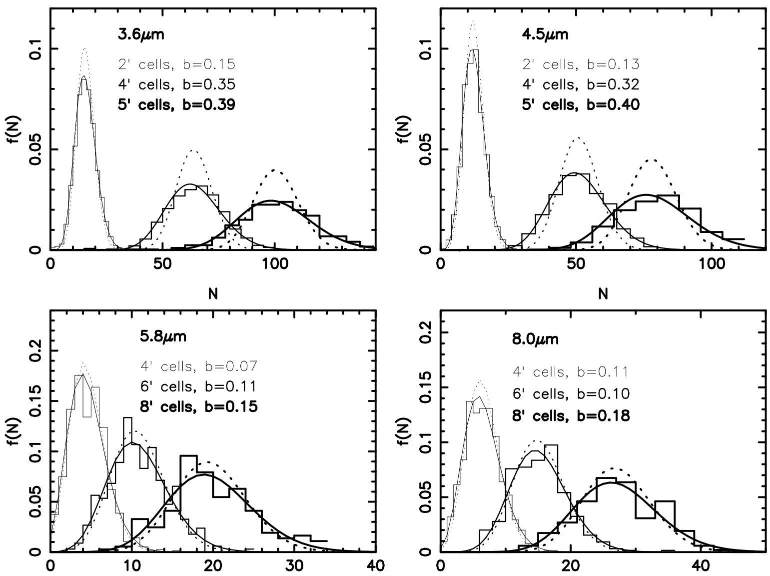


Figure 1. The IRAC sample counts-in-cells histograms (cell sizes increase to the right) and their theoretical descriptions (see text).

A counts-in-cells method is used to establish the probability distribution. In Figure 1 we show our results. Each sample is measured at 3 different scales. For each measurement we plot the fit of the theoretical Gravitational Quasiequilibrium Distribution Function (Saslaw and Hamilton 1984; solid lines) and the Poisson distribution of the same average galaxy count (dotted lines) for comparisons. The deviation from a Poisson distribution, caused by galaxy clustering, is significant at large scales. The gravitational distribution function fits well at all scales. The value of the fitting parameter $b$ shows the strength of clustering. Here $b$ is the ratio of the correlation potential energy and twice the kinetic energy (Saslaw, this volume).

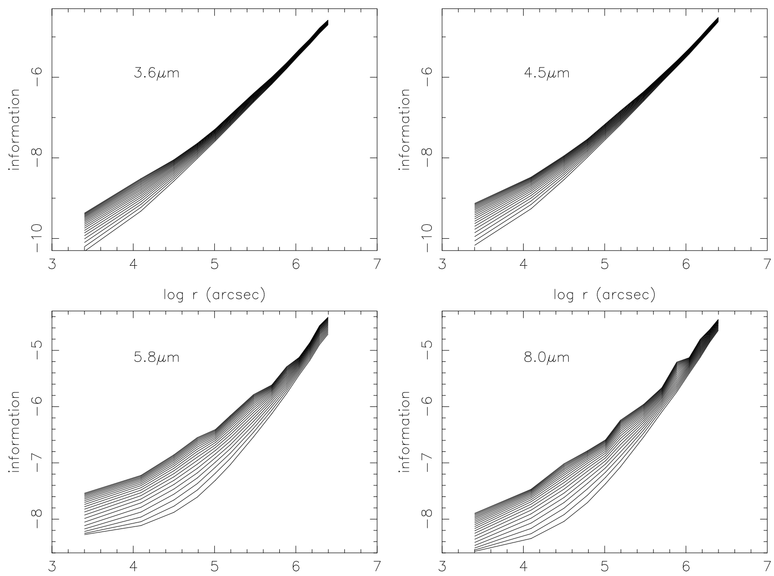


Figure 2. Scaling of Rényi information with cell sizes.

For the first time, we explicitly apply the Rényi Information to describe the galaxy spatial distribution. In Figure 2 we plot the Rényi information measurements, in "bits" of unit log 2, over a range of cell sizes placed on the IRAC samples, and from $\beta = 1$ up to 20 (bottom to top). The crowding of the lines at high $\beta$ indicates that the higher moments can be constrained well from lower moment measurements. The multifractal dimensions can be read from the slopes of these lines. It is significant that these lines are not straight, indicating that the multifractal properties of galaxy distribution vary with scale. The IRAC galaxies appear to occupy space compactly at small scales, and distribute more uniformly at large scales with the fractal dimensions approaching 2. Our flux-limited 2-dimensional samples contain degenerate galaxy evolution in the third dimension. It would be compelling if the scale-dependency of multifractal dimensions is also found in the 3-dimensional galaxy distribution, which may be indications of different phases of the gravitational relaxation.

Using an average measure over the scales, the multifractal dimensions of increasing information orders decrease and approach a limit, the so-called "structure function". This is illustrated in the left panel of Figure 3. The right panel shows the spectra of the multifractal scaling index, related to multifractal dimensions by a Legendre transformation. As the spectra go to zero, the scaling

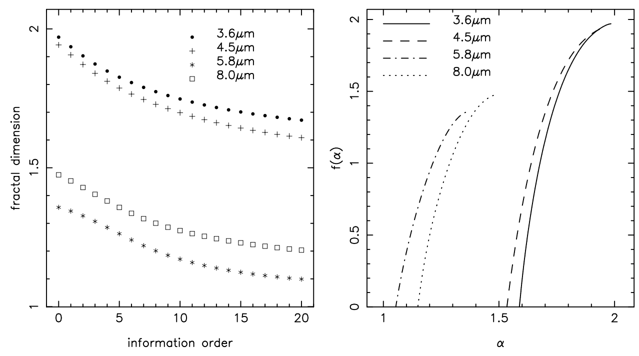


Figure 3. Left panel: multifractal dimensions versus information order. Right panel: the spectra of the multifractal scaling index.

index measures the finite value of the multifractal dimension at information order of infinity. In a different form the spectra contain a statistically complete description of the probability distribution.

## 3. Implications on Computer Simulations

A multifractal can be most straightforwardly simulated by a multiplicative cascade process. The infinite number of statistical moments does not necessarily imply infinite number of parameters for a simulation, if the structure function is known for a given generator of the multiplicative process. For example such function exists for generators of a class of extremal Lévy stable distributions (Schertzer and Lovejoy 1987). The stable distributions are interesting as they replace Gaussian in the generalized Central Limit Theorem if the variances of the component distributions are not finite. However, the scale-dependency of the galaxy multifractality seems to imply that no distribution, stable or not, with a single set of parameters can generate a galaxy spatial distribution in a multiplicative cascade process. Nevertheless, such a process is less expensive than N-body simulation, and explorations of the kind are worthwhile.

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