Post-Newtonian dynamics for orbiting compact objects

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Aim & Outline

- **AIM:** To introduce N-Body astrophysicists to post-Newtonian (PN) approximation, useful to describe the orbiting dynamics of compact objects

Outline

- PN approach to General Relativity; why it is interesting ? & what can it provide ..
- Symbolic demonstration of a PN-computation
- Solving PN-accurate dynamics: subtitles
- Going beyond PN-accurate dynamics & 3-Body interactions

Motivations:I

- Coalescing black-hole binaries have become the focus of a large number of astrophysically & theoretically motivated analytic, semi-analytic & numerical investigations

- Most promising sources of Gravitational Waves (GWs) for LIGO/VIRGO

  GW astronomy with LISA/ SKA will require GWs from Coalescing Massive Black holes.

- Recoiling black holes associated with binary black hole merger should have observational & cosmological consequences

- We need detailed analysis of the dynamics of compact binaries in GR

PN approximation:I

- In General Relativity (GR), zeroth order approximation gives Newtonian gravity

- nPN order: corrections of order

$$\left(\frac{v}{c}\right)^{2n} \sim \left(\frac{Gm}{rc^2}\right)^{n}$$

  to the Newtonian gravity

  m, v & r denote total mass, orbital velocity & separation

- Black holes & neutron stars are modeled as point particles

PN approximation:II

- In the case of non-spinning compact binaries, for LIGO/VIRGO applications, one needs to tackle two problems (usually analyzed separately)
- Problem of finding equations of motion $\ddot{X}$
- Problem of computing gravitational-wave luminosity $\mathcal{L}$, $h_{\times,+}$

$\ddot{X}$	N	1PN	2PN	2.5PN	3PN	3.5PN	4PN	4.5PN	5PN	5.5PN	6PN
$\mathcal{L}$	—	—	—	N	—	1PN	1.5PN	2PN	2.5PN	3PN	3.5PN
$h_{\times,+}$	—	—	N	0.5N	1PN	1.5PN	2PN	2.5PN	3PN		

Motivations:II

- The *response function* of the laser-interferometric detector to gravitational waves from coalescing compact binary <u>in circular orbits</u>:

$$h(t) \equiv \Delta L/L = \frac{C}{d}\left(\dot{\phi}(t)\right)^{2/3} \sin\left(2\phi(t) + \alpha\right), \tag{1}$$

  d is the distance of the binary to the Earth; C and α are some constants,

  $\phi(t)$ is the orbital phase of the binary & $\left(\dot{\phi}(t) \equiv d\phi(t)\Big/ dt\right)$.

- The secular orbital phase $\phi = \phi(t)$ is computed from the balance equation $\frac{dE}{dt} = -\mathcal{L}$

- The expression for $h(t)$ in terms of ϕ & $\dot{\phi}$ requires $h_{\times,+}$

Motivations:III

- GWs from ICBs are being searched by *Matched Filtering* theoretical template sets against the output of LIGO/VIRGO

- Theoretical templates should match the expected (& weak) inspiral signals to within a fraction of a GW cycle in the sensitive bandwidth

- This requires inclusion of higher order PN terms in the evolution of $\phi(t)$ appearing in $h(t)$

Motivations:IV

- The accumulated number of gravitational-wave cycles, $\mathcal{N}_{\mathrm{GW}} \equiv \int (f/\dot{f}) df$, at a PN order in a LIGO/VIRGO-type detector
  Initial & final values of f are 10 Hz & $1/(6^{3/2}\pi m)$ Hz

- It clearly shows 2PN is NOT sufficient

- Are we justified to use PN approximation when $r/m \sim 6$?

- We can not treat ICBs as test-particle in a Sch. BH space-time

	$2 \times 1.4 M_\odot$	$10 M_\odot + 1.4 M_\odot$	$2 \times 10 M_\odot$
Newtonian	16,050	3580	600
First PN	439(104)	212(26)	59(14)
Tail (1.5PN)	-208	-180	-51
Second PN	9(3)	10(2)	4(1)

PN quantities

For comparable mass non-spinning compact binaries in circular orbits, following quantities are available to 3/3.5PN order
they are sufficient to describe accurately the inspiral regime

- 3PN accurate dynamical (orbital) energy $\mathcal{E}(x)$ as a PN series in $x = \left(G\, m\, \omega_{\rm 3PN}/c^3\right)^{2/3}$
  $\omega_{\rm 3PN}(t)$ the 3PN accurate orbital angular frequency
  Damour, Jaranowski & Schäfer (2001)

- 3.5PN accurate expression for GW energy luminosity $\mathcal{L}(x)$
  Blanchet et.al (2002) & (2005)

- 3PN amplitude corrected expressions for $h_+(t)$ & $h_\times(t)$ in terms of the orbital phase $\phi(t)$ and $x(t)$
  Blanchet et.al (2008)

- Approximation techniques that describe inspiralling compact binaries usually require that $(v/c)^2 \sim (\frac{G\,m}{c^2\,r})$ [slow-motion & weak-fields]

Point particles

- **Effacement property** allows to reduce the problem of motion of centers of mass of N bodies to the problem of motion of N point-masses [Arguments due to T. Damour]

- Ellipticity due to tidal distortions arising from Gravitational interactions

$$\epsilon \sim \frac{\left(\frac{G\,M\,L}{R^3}\right)}{\left(\frac{G\,M}{L^2}\right)} \sim \left(\frac{L}{R}\right)^3 \tag{2}$$

Tidal quadrupole moments $\sim \epsilon\,M\,L^2$ & structure dependent interbody forces $\mathcal{F}_{\rm St} \sim G\epsilon\,M^2\,L^2/R^4$

$$\frac{\mathcal{F}_{\rm St}}{\mathcal{F}_{\rm N}} \sim \mathcal{F}_{\rm St} \Big/ \frac{G\,M^2}{R^2} \sim \epsilon\,L^2/R^2 \sim (L/R)^5 \tag{3}$$

For compact objects, $L \sim G\,M/c^2$

- This $\rightarrow \frac{\mathcal{F}_{\rm St}}{\mathcal{F}_{\rm N}} \sim \left(\frac{G\,M}{c^2\,R}\right)^5$; 5PN order ...

How to get Newtonian dynamics from GR

- For a slowly moving test-particle in quasi-stationary & weak gravitational field

$$g_{\mu\,\nu} = \eta_{\mu\,\nu} + h_{\mu\,\nu}\,, |h_{\mu\,\nu} \ll 1| \tag{4}$$

The geodesic equation $\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\,\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$ becomes

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\alpha\,\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{5}$$

In our case $\frac{dx^0}{d\tau} \sim 1$ & $\frac{dx^i}{d\tau} \sim 0$

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{0\,0} \sim -\left(\frac{1}{2}h_{0\,0\,,i} - h_{0\,i\,,0}\right) \tag{6}$$

$$\frac{d^2x^i}{dt^2} \sim \frac{1}{2}h_{0\,0\,,i} \to \ddot{\mathbf{x}} = -\nabla\phi \tag{7}$$

This $=>$ $g_{0\,0} = -(1 + 2\,\phi)$

Quantities derivable from PN-accurate dynamics

PN-accurate compact binary dynamics:I

- By iterating Einstein's field equations,in principle, it is possible to compute 3PN-accurate Lagrangian

$$L^{\mathrm{harmonic}} \equiv L[\mathbf{y}_A(t), \mathbf{v}_A(t), \mathbf{a}_A(t)] \text{ ,A=1,2\&i=1,2,3)} \tag{8}$$

  instantaneous positions $y_A^i(t) \equiv \mathbf{y}_A(t)$
  coordinate velocities $v_A^i(t) \equiv \mathbf{v}_A(t) = d\mathbf{y}_A/dt$,
  coordinate accelerations $a_A^i(t) \equiv \mathbf{a}_A(t) = d\mathbf{v}_A/dt$.

- The explicit derivation of 1PN-accurate Lagrangian will be demonstrated later..

- It provides a number of useful quantities ..

- We neglect the effects of radiation reaction..

PN-accurate compact binary dynamics:II

- To 1PN accuracy $1 \longleftrightarrow 2$

$$L = \frac{Gm_1m_2}{2r_{12}} + \frac{m_1v_1^2}{2} + \frac{1}{c^2}\left\{ -\frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{m_1v_1^4}{8} + \frac{Gm_1m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2)\right)\right\} \quad (9)$$

$\mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12}$, and the scalar products are written e.g. $(n_{12}v_2) = \mathbf{n}_{12}.\mathbf{v}_2$.

-

$$E = \frac{m_1v_1^2}{2} - \frac{Gm_1m_2}{2r_{12}} + \frac{1}{c^2}\left\{ \frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{3m_1v_1^4}{8} + \frac{Gm_1m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2)\right)\right\} \quad (10)$$

PN-accurate compact binary dynamics:III

- 1PN-accurate linear momentum

$$
\begin{aligned}
P^i &= m_1 v_1^i + \frac{1}{c^2}\Bigg\{ - n_{12}^i \frac{G m_1 m_2}{2 r_{12}} (n_{12} v_1) \\
&+ v_1^i \left(- \frac{G m_1 m_2}{2 r_{12}} + \frac{m_1 v_1^2}{2} \right) \Bigg\} \qquad (11)
\end{aligned}
$$

- 1PN-accurate angular momentum

$$
\begin{aligned}
J^i &= \varepsilon_{ijk} m_1 y_1^j v_1^k + \frac{1}{c^2} \varepsilon_{ijk} \Bigg\{ y_1^j v_1^k \left(\frac{3 G m_1 m_2}{r_{12}} + \frac{m_1 v_1^2}{2} \right) \\
&- y_1^j v_2^k \frac{7 G m_1 m_2}{2 r_{12}} + y_1^j y_2^k \frac{G m_1 m_2}{2 r_{12}^2} (n_{12} v_1) \Bigg\} \qquad (12)
\end{aligned}
$$

- 1PN-accurate COM integral $G^i = P^i t + K^i$

$$
G^i = m_1 y_1^i + \frac{1}{c^2} \Bigg\{ y_1^i \left(- \frac{G m_1 m_2}{2 r_{12}} + \frac{m_1 v_1^2}{2} \right) \Bigg\} \qquad (13)
$$

PN-accurate compact binary dynamics:IV

- In the COM frame $\boldsymbol{K} \equiv \boldsymbol{P} \equiv 0$,

$$
\begin{aligned}
y_1^i &= \eta\, y^i + \frac{\eta\,\delta}{2\,c^2}\left\{ v^2 - \frac{G\,m}{r} \right\} y^i \\
y_2^i &= -\eta\, y^i + \frac{\eta\,\delta}{2\,c^2}\left\{ v^2 - \frac{G\,m}{r} \right\} y^i
\end{aligned}
\tag{14}
$$

$\eta = \mu/m$; $\delta = (m_1 - m_2)/m$.

-

$$
\dot{\mathbf{v}} \equiv \dot{\mathbf{v}}_{\mathbf{1}} - \dot{\mathbf{v}}_{\mathbf{2}} = -\frac{G\,m}{r^2}\,\mathbf{n} + \frac{G\,m}{c^2\,r}\left\{ \Big(... \Big)\,\mathbf{n} + \Big(... \Big)\,\mathbf{v} \right\}
\tag{15}
$$

For general orbits, we know $\dot{\mathbf{v}}$ to $\mathcal{O}(1/c^7)$ & conserved quantities to $\mathcal{O}(1/c^6)$ order

More details in Andrade, Blanchet & Faye, gr-qc/0011063

Usable 3.5PN $\dot{\mathbf{v}}, \mathcal{E} \& \mathcal{J}^i$

- $\ddot{\mathbf{x}}$ for non-spinning comparable mass compact binaries to 3.5PN order in *Eqs. (2.7),(2.8) & (2.9) in T. Mora & C. M. Will, Phys. Rev. D* **69***, 104021 (2004)*

$$
\begin{aligned}
\mathbf{a} \quad &\equiv \frac{d^2\mathbf{x}}{dt^2} = -\left(\frac{m}{r^3}\right)\left[\left(1+A\right)\mathbf{n} + B\,\mathbf{v}\right] \\
A &= A_1(r,\dot{r},\dot{\phi},m,\eta) + A_2(r,\dot{r},\dot{\phi},m,\eta) + A_{2.5}(r,\dot{r},\dot{\phi},m,\eta) \\
&+ A_3(r,\dot{r},\dot{\phi},m,\eta) + A_{3.5}(r,\dot{r},\dot{\phi},m,\eta) \\
&\quad \text{Similar expressions for B}
\end{aligned}
\tag{16}
$$

- 3PN-accurate conserved orbital energy & angular momentum are given by Eq. (2.11) & (2.12)

$$
\begin{aligned}
\mathcal{E} &= \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 \\
\mathcal{E}_0 &= \mu\left(\frac{v^2}{2} - \frac{m}{r}\right), \\
\mathbf{J} &= \mathbf{J}_0 + \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 \\
\mathbf{J}_0 &= \mu\,\mathbf{r}\times\mathbf{v}
\end{aligned}
\tag{17}
$$

$h_{\times,+}$:I

- GW polarizations $h_{\times,+}$ are defined

$$h_+ \;=\; \frac{1}{2}\left(p_i\, p_j - q_i\, q_j\right) h_{ij}^{TT} \,, \tag{18}$$

h_{ij}^{TT}, the transverse-traceless (TT) part of the radiation field

$$h_{ij}^{TT} = \frac{2G}{c^4 r'}\left\{ I_{ij}^{(2)} + \frac{1}{c}\left[\frac{1}{3} n'_a I_{ija}^{(3)} + \frac{4}{3}\varepsilon_{ab(i} J_{j)a}^{(2)} n'_b\right] + \ldots\ldots \right\}^{\mathrm{TT}} \tag{19}$$

$A_{TT}^{ij} = A^{lm}(P^{il}P^{jm} - \frac{1}{2}P^{ij}P^{lm})$, where $P^{ij} = \delta^{ij} - n'^i n'^j$; n'^i: unit vector from source to observer

I^{ij}; PN-accurate mass quadrupole moment & J^{ij}; current quadrupole moment

- h_{ij}^{TT} is expressible in terms of STF multipoles of source densities; usually computed via Blanchet-Damour-Iyer formalism

$h_{\times,+}$:II

- h_{ij}^{TT} analogues to A_j appearing in electromagnetism;

$$A_j = \frac{1}{c\,r'} d_j^{\mathrm{T}}; \quad d_j^{\mathrm{T}} \equiv P_{jk}\, d_k$$

$$\mathcal{L}_{\mathrm{em}} \propto r'^2 \int \left(\dot{A}_j \dot{A}_j\right) d\Omega(\mathbf{n}') \rightarrow \frac{1}{c^3} \ddot{d}_j^{\mathrm{T}} \ddot{d}_j^{\mathrm{T}} \tag{20}$$

- However, in GR, there are NO ‘mass‘ dipole & ‘mass’ magnetic dipole radiations as total P^i & J^i are conserved

$$\ddot{d}_j = \sum_A m_A \ddot{x}_j^A = \sum_A p_j^A \rightarrow 0 \tag{21}$$

‘mass ’ magnetic dipole

$$\mu_i \propto \varepsilon_{ijk} \sum_A x_A^j \left(m_A v_A^k\right) = \sum_A J_j^A \rightarrow 0 \tag{22}$$

Therefore, the lowest-order radiation in GR is quadrupolar ...

$h_{\times,+}$:III

- The radiate energy loss (-ve of the GW luminosity) can be computed

$$
\begin{aligned}
\left(\frac{dE}{dt}\right)_{\mathrm{FZ}} &= -\frac{c^3\, r'^2}{32\pi G}\int \left(\dot{h}^{TT}_{km}\dot{h}^{TT}_{km}\right) d\Omega(\mathbf{n}') \\
&= -\frac{G}{c^5}\left\{\frac{1}{5} I^{(3)}_{ij} I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189} I^{(4)}_{ijk} I^{(4)}_{ijk} + \frac{16}{45} J^{(3)}_{ij} J^{(3)}_{ij}\right]\right. \\
&+ \left.\mathcal{O}(c^{-3}\right\} \qquad (23)
\end{aligned}
$$

- For circular inspiral, the above quantity is computed to 3.5PN order [neglected terms are $\mathcal{O}(c^{-8}$ order in the above Eq.]

- 3PN accurate orbital energy, 3.5PN accurate L & 3PN accurate expressions for $h_{\times,+}$ are the crucial quantities to do astrophysics with eventual GW observations of inspiralling compact binaries

GW search templates

PN quantities

Blanchet, Damour, Schäfer & their collaborators, after many years of computations, provided FOUR valuable expressions for compact binaries in PN accurate circular orbits

- 3PN accurate dynamical (orbital) energy $\mathcal{E}(x)$ as a PN series in $x = (G\, m\, \omega_{\rm 3PN}/c^3)^{2/3}$
  $\omega_{\rm 3PN}(t)$ the 3PN accurate orbital angular frequency
  Damour, Jaranowski & Schäfer (2001)

- 3.5PN accurate expression for GW energy luminosity $\mathcal{L}(x)$
  Blanchet et.al (2002) & (2005)

- 3PN amplitude corrected expressions for $h_+(t)$ & $h_\times(t)$ in t erms of the orbital phase $\phi(t)$ and $x(t)$
  Blanchet et.al (2008)

LAL Routines

The LSC Algorithms Library (LAL) employs these inputs to construct various types of search templates

TaylorT1 Damour, Iyer & Sathyaprakash (2001)

$$h(t) \propto \left(\frac{G\,m\,\omega(t)}{c^3} \right)^{2/3} \cos 2\,\phi(t)\,, \tag{24}$$

PN-accurate LAL templates require that inspiral is along *exact* circular orbits !!

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TaylorT1 $h(t)$

$$\frac{d\phi(t)}{dt} = \omega(t)\,; \quad \frac{d\,\omega(t)}{dt} = -\mathcal{L}(\omega) \Big/ \frac{d\mathcal{E}}{d\omega}\,, \tag{25}$$

To compute TaylorT1 3.5PN $h(t)$, one needs 3.5PN accurate GW luminosity $\mathcal{L}(\omega)$ & 3PN orbital energy $\mathcal{E}$

LAL Routines

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TaylorT1 Damour, Iyer & Sathyaprakash (2001)

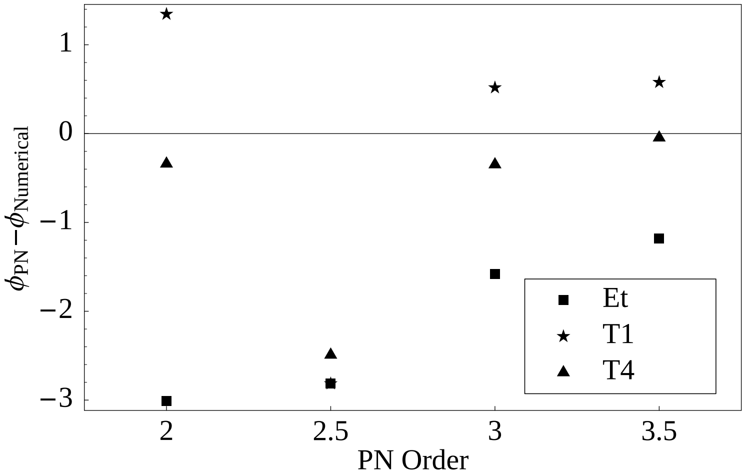
$$h(t) \propto \left(\frac{G\,m\,\omega(t)}{c^3}\right)^{2/3} \cos 2\,\phi(t)\,, \tag{24}$$

PN-accurate LAL templates require that inspiral is along *exact* circular orbits !!

TaylorT4 3.5PN $h(t)$: Very close to NR inspiral $h(t)$, but not in LAL

$$\frac{d\phi(t)}{dt} \equiv \omega(t)\,;\ \frac{d\,\omega(t)}{dt} = \frac{96}{5}\left(\frac{G\,\mathcal{M}\,\omega}{c^3}\right)^{5/3}\omega^2 \Big\{ 1 + \mathcal{O}(\nu) + \mathcal{O}(\nu^{3/2}) + \mathcal{O}(\nu^2) + \mathcal{O}(\nu^{5/2}) + \mathcal{O}(\nu^3) + \mathcal{O}(\nu^{7/2}) \Big\}\,, \tag{25}$$

GW phase evolution: PN Vs NR



TaylorEt $h(t)$

The restricted 3.5PN accurate TaylorEt $h(t)$ is given by

$h(\hat{t}) \propto \tilde{\mathcal{E}}(\hat{t}) \cos 2\,\phi(\hat{t})$

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$h(\hat{t}) \propto \tilde{\mathcal{E}}(\hat{t}) \cos 2\,\phi(\hat{t})$

$$\frac{d\phi}{d\hat{t}} = \tilde{\mathcal{E}}^{3/2} \left\{ 1 + \tilde{\mathcal{E}} \left[.. \right] + \tilde{\mathcal{E}}^{2} \left[.. \right] \right\}, \qquad (26)$$

$$\frac{d\tilde{\mathcal{E}}}{d\hat{t}} = \frac{64}{5}\,\eta\,\tilde{\mathcal{E}}^{5} \left\{ 1 + \tilde{\mathcal{E}} \left[.. \right] + ...\tilde{\mathcal{E}}^{7/2} \left[.. \right] \right\},$$

$\tilde{\mathcal{E}}/2$ is the dimensionless non-relativistic energy per unit reduced mass

GW driven inspiral is along PN-accurate circular orbit...

Spin effects: a primer

Kidder, PRD,bf 52, 821, 1995; Blanchet, Buonanno & Faye, gr-qc/0605140, gr-qc/0605139 & recent papers from Jena

Including spin effects:I

- The dominant spin effect is that due to the relativistic spin-orbit coupling
  Its contribution to reduced Hamiltonian $\mathcal{H} = H/\mu$

  $$\mathcal{H}_{\rm SO} \sim \frac{\boldsymbol{L}\cdot\boldsymbol{S_1}}{c^2\,r^3} = \mu\,\frac{(\boldsymbol{r}\times\boldsymbol{v})\cdot\boldsymbol{S_1}}{c^2\,r^3} \qquad (27)$$

  This is formerly at 1PN order; BUT

  $S_1 \sim m_{\rm co}\, r_{\rm co}\, v^{\rm spin}$ & for compact objects $r_{\rm co} \sim \frac{G\,m_{\rm co}}{c^2} \rightarrow \mathcal{H}_{\rm SO}$ at 2PN order, if $v^{\rm spin} < c$. However, if $v^{\rm spin} = c$, $\mathcal{H}_{\rm SO}$ contributions stand at 1.5PN order

- Relativistic spin-orbit coupling provides corrections to $\ddot{\boldsymbol{X}}$ at 2PN/1.5PN order order along with expressions for $\dot{\boldsymbol{S}}_{\boldsymbol{1}}, \dot{\boldsymbol{S}}_{\boldsymbol{2}}, \& \dot{\boldsymbol{L}}$

  Its contributions to far-zone GW luminosity is also available

Including spin effects:II

- Recently, next to leading order corrections to $\ddot{\boldsymbol{X}}, \dot{\boldsymbol{S}}_1, \dot{\boldsymbol{S}}_2, \& \dot{\boldsymbol{L}}$ and far-zone GW luminosity due to spinning point-particles were obtained
- Black hole absorption occurs at this PN order for the first time ...

- If both compact objects are spinning, spin-spin interactions are important & its dominant contributions

$$\mathcal{H}_{\rm SS} \sim \frac{\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + (\boldsymbol{n} \cdot \boldsymbol{S}_1 + \boldsymbol{n} \cdot \boldsymbol{S}_1)}{c^2\, r^3} \tag{28}$$

$\boldsymbol{S}_1 \sim \frac{G\, m_{\rm co}}{c^2} \rightarrow$ the dominant $\mathcal{H}_{\rm SS}$ appear at 2PN/3PN order..

- Explicit contributions to $\ddot{\boldsymbol{X}}, \dot{\boldsymbol{S}}_1, \dot{\boldsymbol{S}}_2, \& \dot{\boldsymbol{L}}$ and far-zone GW luminosity due to spinning point-particles were obtained.

- Next-to-leading order contribution to $\mathcal{H}_{\rm SS}$ are conceptually & computationally difficult to compute .. ongoing efforts ..

Recoil: a primer

Blanchet, Qusailah& Will, astro-ph/0507692 & ....

GW induced recoil:I

- If gravitational radiation field created by a compact binary is asymmetric, GW induced recoil occurs
- $m_1 \neq m_2 \& \mathbf{S_1} \equiv \mathbf{S_2} = 0;$

  $m_1 = m_2; \mathbf{S_1} \& \mathbf{S_2} \neq 0$

  $m_1 \& m_2 \neq 0; \mathbf{S_1} \& \mathbf{S_2} \neq 0$
- Net recoil is due to GW induced damping
- To get lowest order asymmetric radiation field, we need to beat the mass quadrupole with mass octupole and current quadrupole moments.
- Higher order moments $\rightarrow$ a tiny effect

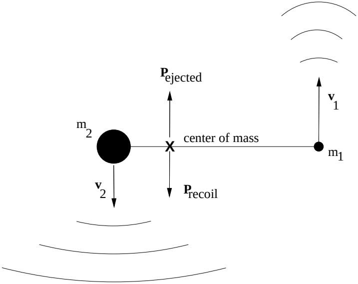
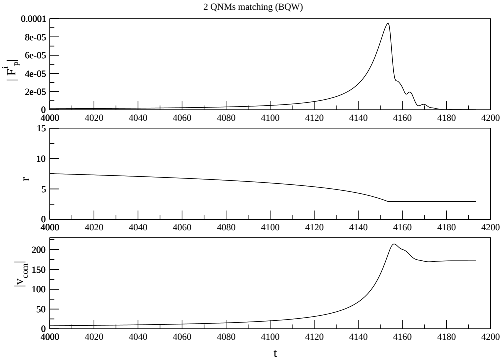


Figure from Wiseman's paper (1992)

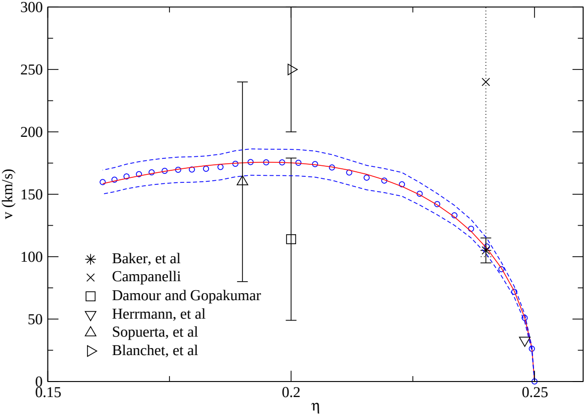
GW induced recoil!!



- Various aspects of GW induced recoil during BBH coalescence based on EOB approach & for non-spinning BHs

- Maximum recoil during the merger phase

- Recoil estimates based on numerical relativity is consistent with these observations



300
250
200
150
100
50
0
0.15
0.2
0.25
η
v (km/s)
Baker, et al
Campanelli
Damour and Gopakumar
Herrmann, et al
Sopuerta, et al
Blanchet, et al

GW induced recoil:IV

- In PN approximation, Linear momentum flux associated with the anisotropic emission of GWs

$$\mathcal{F}^i_{\mathbf{P}} = -\frac{G}{c^7}\left\{\frac{2}{63}\, I^{(4)}_{ijk}\, I^{(3)}_{jk} + \frac{16}{45}\,\varepsilon_{ijk}\, I^{(3)}_{jl}\, J^{(3)}_{kl} + \mathcal{O}(c^{-2})\right\} \quad (29)$$

J_{kl} & I_{ijk} are $\propto (1 - 4\,\eta)^{1/2}$

- To compute the GW induced recoil, one invokes momentum balance argument

$$\frac{dP^i}{dt} = -\mathcal{F}^i_{\mathbf{P}}(t)\,, \;\rightarrow\; \Delta P^i(t) = -\int_{-\infty}^{t} dt\, \mathcal{F}^i_{\mathbf{P}}(t)$$
$$V^i = \Delta P^i/\sqrt{m^2 + \Delta \mathbf{P}^2} \;\rightarrow\; V^i \sim \Delta P^i/m \quad (30)$$
$$V^i = \frac{464}{105}\,\eta^2\,\frac{\delta m}{m}\,x^4\,n^i$$

For circular inspiral, $\mathcal{F}^i_{\mathbf{P}}$ is known up to 2PN order ...

- Numerical Relativity recoil estimates follows a similar procedure ..

Symbolic demonstration : How to get 1PN-accurate $L(\boldsymbol{y}, \boldsymbol{v})$

Text-books by MTW, N. Straumann, Will

Demo:I

- Aim is to solve $R_{\mu\,\nu} = T_{\mu\,\nu} - \frac{1}{2}\,g_{\mu\,\nu}\,T^{\xi}_{\xi}$ in a perturbative manner & let $\chi \sim \left(\frac{G\,m}{c^2\,r}\right)^{1/2} \sim \frac{v}{c}$
- We are only interested to χ^2 order & $\rightarrow$

$$g_{0\,0} = -1 +^{(2)} g_{00} +^{(4)} g_{00}\,,\; g_{0\,i} =^{(3)} g_{0i}\,,\;\; g_{ij} = \delta_{ij} +^{(2)} g_{ij} \tag{31}$$

- From $g_{\mu\,\xi} g^{\nu\,\xi} = \delta^{\nu}_{\mu}$, we infer

$$^{(2)}g^{00} = -^{(2)}g_{00}\,,\; ^{(2)}g^{ij} = -^{(2)}g_{ij}\,,\; ^{(3)}g^{0i} =^{(3)} g_{0i}\,, \tag{32}$$

- Compute Christoffel symbols & the components of the Ricci tensor

$$R_{0\,0} =^{(2)} R_{00} +^{(4)} R_{00}\,,\; R_{0\,i} =^{(3)} R_{0i}\,,\;\; R_{ij} =^{(2)} R_{ij} \tag{33}$$

- It is straightforward to obtain above quantities in terms of $^{(n)}g_{\mu\,\nu}$, their spatial & time derivatives

Demo:II

- In order to make RHS of ${}^{(2)}R_{00}$, ${}^{(4)}R_{00}$, ${}^{(3)}R_{0i}$, & ${}^{(2)}R_{ij}$ less complicated, we apply certain gauge-condition

- This leads to

$$\begin{aligned} {}^{(2)}R_{00} &\sim \triangle^{(2)}g_{00}\,,\ {}^{(2)}R_{ij} \sim \triangle^{(2)}g_{ij} \\ {}^{(4)}R_{00} &\sim \left(\triangle^{(4)}g_{00},{}^{(2)}g_{ij},{}^{(2)}g_{00,ij},\ldots\right) \\ {}^{(3)}R_{0i} &\sim \left(\triangle^{(3)}g_{0i},{}^{(2)}g_{ij,0j},\ldots\right) \end{aligned} \tag{34}$$

- Apply similar PN ansatz to $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\xi}_{\xi}$;

  $T^{\mu\nu}$ & $g^{\mu\nu}$ have similar PN expansions & this leads to

$${}^{(0)}S_{00} \sim {}^{(2)}T^{00}\,,\ {}^{(0)}S_{ij} \sim \delta_{ij}{}^{(2)}T^{00},\ \ {}^{(2)}S_{00} \sim (\ldots),\ \ {}^{(1)}S_{0i} \sim (\ldots) \tag{35}$$

Demo:III

- $$\triangle^{(2)}g_{00} \sim^{(2)} T^{00}\,,\ \triangle^{(2)}g_{ij} \sim \delta_{ij}\,{}^{(2)}T^{00}\,, \tag{36}$$

  $$^{(2)}g_{00} \sim \phi\,,\ ^{(2)}g_{ij} \sim \delta_{ij}\,\phi\,,\ \ \phi \sim \int d^3x'\,\frac{^{(2)}T^{00}(t,\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{37}$$

- Using above expressions, somewhat complicated partial differential Eqs for $^{(4)}g_{00}$ and $^{(3)}g_{0i}$ can simplified ..

  $$^{(4)}g_{00} \sim \phi^2 + \psi\,,\ \triangle\psi \sim \left(^{(2)}T^{00} +^{(2)} T^{ii}\right) \tag{38}$$

- To solve the Eq for $^{(3)}g_{0i}$, introduce two more potentials

  $$\xi_i \sim \int d^3x'\,\frac{^{(1)}T^{0i}(t,\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}'|}\,,\ \chi \sim \int d^3x' |\boldsymbol{x}-\boldsymbol{x}'|\,{}^{(2)}T^{00}(t,\boldsymbol{x}) \to \triangle\chi = \phi \tag{39}$$

  Finally $^{(3)}g_{0i} \sim \xi_i + \chi_{i0}$

Demo:IV

- The gauge condition introduced implies that, in general, $T^{\alpha\,\beta}_{;\beta} \equiv 0$ However, at this order, the potentials introduced do satisfy the above relation...

- To get 1PN-accurate Lagrangian, we compute the EOM of a particle in an external gravitational field, defined by $(\phi, \psi, \chi, \xi_i)$

$$\delta \int dt \left(\frac{d\tau}{dt}\right) = 0\,, \quad \left(\frac{d\tau}{dt}\right)^2 = -g_{\mu\,\nu}\,\frac{dx^\mu}{dt}\,\frac{dx^\nu}{dt}$$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \boldsymbol{v}^2 -^{(2)} g_{00} -^{(4)} g_{00} - 2\,^{(3)}g_{0i}\,v^i -^{(2)} g_{ij}\,v^i\,v^j \qquad (40)$$

- PN-accurate Lagrangian $L := 1 - \frac{d\tau}{dt}$ & hence expressible in terms of $(\phi, \psi, \chi, \xi_i)$, their spatial & temporal derivatives..

Demo:IV

- We are interested in point-particles & therefore heuristically introduce

$$T^{\mu\nu}(\mathbf{x},t) = \sum_{A=1}^{N} m_A \frac{dy_A^\mu}{dt}\frac{dy_A^\nu}{dt}\frac{1}{\sqrt{-g}}\frac{dt}{d\tau}\delta(\mathbf{x}-\mathbf{y}_A(t)) \ , \tag{41}$$

- This makes the evaluation of potentials fast & easy
  For example, $\chi \sim \sum_a m_a |\boldsymbol{x} - \boldsymbol{x_a}|$, & $\xi_i \sim \sum_a \frac{m_a v_a^i}{|\boldsymbol{x}-\boldsymbol{x_a}|}$ ...
- The Lagrangian L_a of a particle a in the gravitational field of other particles & represented by $(\phi, \psi, \chi, \xi_i)$ follows ...
- The total Lagrangian should satisfy $Limit_{\ m_a \to 0} \frac{L}{m_a} = L_a \to L = \sum_a m_a\ L_a$

$$\begin{aligned} L &= \frac{Gm_1m_2}{2r_{12}} + \frac{m_1v_1^2}{2} + \frac{1}{c^2}\Bigg\{ -\frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{m_1v_1^4}{8} \\ &+ \frac{Gm_1m_2}{r_{12}}\Big(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2)\Big)\Bigg\} \end{aligned} \tag{42}$$

$1 \longleftrightarrow 2$

Solving PN-accurate orbital dynamics

Papers from Jena during 2004-2007

PN-accurate solving:I

- Numerical solution of PN-accurate $\ddot{\mathbf{x}}$ requires
  $$d\mathbf{x}/dt \equiv \mathbf{v}; d\mathbf{v}/dt \equiv \ddot{\mathbf{x}} = \mathcal{A}\left(\left\{r, \dot{r}, \dot{\phi}\right\}\mathbf{x} + \left\{r, \dot{r}, \dot{\phi}\right\}\mathbf{v}\right)$$

- Reactive contributions to $\ddot{\mathbf{x}}$ is known only to relative 1PN order; but far-zone fluxes are known to higher PN orders..

- Numerical solution of $\ddot{\mathbf{x}}$ need not be PN-accurate & this may lead to undesirable effects

- If $\ddot{\mathbf{x}}$ is employed in an N-Body code, isolating physical effects due to PN-accurate $\ddot{\mathbf{x}}$ for isolated binaries from those due to many-body effects may become demanding ..

- It is desirable to have a semi-analytic prescription to solve $\ddot{\mathbf{x}}$

  It is also required to construct $h_{\times,+}(t)$ associated with compact binaries in inspiralling eccentric orbits..

PN-accurate Solving:II

- Let the relative acceleration of the compact binary be $\ddot{\mathbf{x}} \equiv \mathcal{A} = \mathcal{A}_0 + \mathcal{A}'$.

- $\mathcal{A}_0$ **is the 'conservative' (integrable) part &** $\mathcal{A}'$ **is the reactive perturbative part.**

- The method first constructs the solution to the 'unperturbed' system, whose dynamics is governed by $\mathcal{A}_0$.

- **The solution to the binary dynamics, governed by** $\mathcal{A}$**, is obtained by *varying the constants* in the generic solutions of the unperturbed system.**

PN-accurate Solving:III

- Recall that for non-spinning compact binaries, conservative part is available to 3PN order

- **For 3PN-accurate dynamics, in the COM frame, there are 4 first integrals.**
  The 2PN accurate energy and angular momentum of the binary, denoted by c_1 & c_2^i:

$$
\begin{aligned}
c_1 &= \mathcal{E}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{v_1}, \mathbf{v_2})|_{\text{2PN CM}} \,, \\
c_2^i &= \mathcal{J}_i(\mathbf{x_1}, \mathbf{x_2}, \mathbf{v_1}, \mathbf{v_2})|_{\text{2PN CM}} \,,
\end{aligned}
$$

- The vectorial structure of c_2^i, indicates that the unperturbed motion takes place in a plane.

- **Close inspection of 3.5PN- accurate $\mathcal{A} \rightarrow$ similar picture even when radiation reaction is present**

- **We can introduce polar coordinates in the plane of the orbit**

PN-accurate Solving:IV

- **The functional form for the solution to the unperturbed (3PN accurate) equations of motion**

$$r = S(l; c_1, c_2) \quad ; \quad \dot{r} = n \frac{\partial S}{\partial l}(l; c_1, c_2)\,,$$

$$\phi = \lambda + W(l; c_1, c_2) \quad ; \quad \dot{\phi} = (1+k)n + n \frac{\partial W}{\partial l}(l; c_1, c_2)\,,$$

The basic angles l and λ are given by

$$l = n(t - t_0) + c_l\,, \quad \lambda = (1+k)n(t - t_0) + c_\lambda$$

- l, λ & $S(l)$, $W(l)$, $\frac{\partial W}{\partial l}(l)$ are periodic in l with a period of 2π.

- The radial period $n = 2\pi/T_r$ & periastron advance parameter k are gauge invariant functions of c_1 & $c_2 = |c_2^i|$.

- **t_0 is some initial instant and the constants c_l & c_λ, the corresponding values for l & λ.**

PN-accurate Solving:V

- **We construct the solution of the perturbed system, defined by $\mathcal{A}$ in the following way.**

- We keep the same the functional form for r, $\dot{r}$, ϕ & $\dot{\phi}$, as functions of l & λ, but allow temporal variation in $c_1 = c_1(t)$ & $c_2 = c_2(t)$.

- **Also, we have following definitions for l & λ**
  $l \equiv \int_{t_0}^{t} n\,dt \,+\, c_l(t) \qquad \lambda \equiv \int_{t_0}^{t} (1+k)\,n\,dt \,+\, c_\lambda(t)$.

- **Note evolving quantities $c_l(t)$, & $c_\lambda(t)$.**

- The four variables $\{c_1, c_2, c_l, c_\lambda\}$ replace the original four dynamical variables r, $\dot{r}$, ϕ & $\dot{\phi}$ and $\{c_\alpha\}$ satisfies first order evolution equations.

PN-accurate Solving:VI

- **The explicit expressions for** $\{dc_\alpha/dt\}$ **read**

$$\begin{aligned}
\frac{dc_1}{dt} &= \frac{\partial c_1(\mathbf{x},\mathbf{v})}{\partial v^i}\,\mathcal{A}'^i\,, \\
\frac{dc_2}{dt} &= \frac{\partial c_2(\mathbf{x},\mathbf{v})}{\partial v^j}\,\mathcal{A}'^j\,, \\
\frac{dc_l}{dt} &= -\left(\frac{\partial S}{\partial l}\right)^{-1}\left\{\frac{\partial S}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial S}{\partial c_2}\frac{dc_2}{dt}\right\}, \\
\frac{dc_\lambda}{dt} &= -\frac{\partial W}{\partial l}\frac{dc_l}{dt}-\frac{\partial W}{\partial c_1}\frac{dc_1}{dt}-\frac{\partial W}{\partial c_2}\frac{dc_2}{dt}\,.
\end{aligned}$$

- The evolution of Eqs. for c_l & c_λ follow from the fact that we have same functional form for $\dot{r}$ & $\dot{\phi}$ in unperturbed & perturbed cases.

PN-accurate Solving:VII

- $c_\alpha(l) = \bar{c}_\alpha(l) + \tilde{c}_\alpha(l)$; $\bar{c}_\alpha(l) \rightarrow$ a slow secular drift & $\tilde{c}_\alpha(l) \rightarrow$ periodic fast oscillations

- It can be demonstrated that to all PN orders $\frac{dc_\lambda}{dt} \equiv 0 \equiv \frac{dc_l}{dt}$

- Therefore, the reactive secular evolution of a PN-accurate eccentric orbit can be obtained with $\frac{d\bar{c}_1}{dt}$ & $\frac{d\bar{c}_2}{dt}$

- While pursuing GW phasing for eccentric binaries, it is advisable to use $\xi \equiv \frac{-2\,E}{(\mu\,c^2)}$ and e_t as appropriate variables to describe PN-accurate eccentric orbit

- Idea is to impose $\xi(t)$ and $e_t(t)$ on 3PN accurate quasi-Keplerian representation of PN-accurate eccentric orbit that provides explicit PN-accurate semi-analytic solution to conservative 3PN-accurate $\ddot{\mathbf{x}}$

GQKP:I

- 3PN-accurate conservative orbital dynamics of non-spinning compact binaries, either in Lagrangian or Hamiltonian approach, allows 'Keplerian type' parametric solution:

- Keplerian parametric solution for Newtonian-accurate orbital motion
- $\mathbf{r} = r(\cos\varphi, \sin\varphi)$

$$R = a(1 - e\cos u)\,,$$

$$\phi - \phi_0 = v \equiv 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{u}{2}\right], \tag{43}$$

$$l \equiv n(t - t_0) = u - e\,\sin u\,, \tag{44}$$

The orbital elements a, e, n are orbital energy E, & angular momentum L

GQKP:II

- For 3PN-accurate orbital motion, the radial motion is also parametrized by

$$r = a_r\,(1 - e_r \cos u)$$

- a_r, e_r 3PN accurate semi-major axis & 'radial eccentricity'.
- These are expressible in terms of orbital energy E, angular momentum L and m_1 & m_2.
- u is the eccentric anomaly.

GQKP:III

- However, the 3PN-accurate angular motion is given by

$$\varphi - \varphi_0 = (1+k)\,v + \left(\frac{f_{4\varphi}}{c^4} + \frac{f_{6\varphi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\varphi}}{c^4} + \frac{g_{6\varphi}}{c^6}\right)\sin 3v + \frac{i_{6\varphi}}{c^6}\sin 4v + \frac{h_{6\varphi}}{c^6}\sin 5v\,,$$

where $v = 2\arctan\left[\left(\frac{1+e_\varphi}{1-e_\varphi}\right)^{1/2}\tan\frac{u}{2}\right]$ is the true anomaly.

- k measures the advance of the periastron & e_φ is the 'angular eccentricity'
- $f_{4\varphi}$, $f_{6\varphi}$, $g_{4\varphi}$, $g_{6\varphi}$, $i_{6\varphi}$, and $h_{6\varphi}$ are 2PN & 3PN order orbital functions expressible in terms of E, L, m_1 & m_2

GQKP:IV

- The 3PN accurate **'Kepler equation'**, which connects the eccentric anomaly to the coordinate time reads

$$
\begin{aligned}
l \equiv n\,(t - t_0) &= u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6} \right)(v - u) \\
&+ \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6} \right) \sin v + \frac{i_{6t}}{c^6} \sin 2v + \frac{h_{6t}}{c^6} \sin 3v
\end{aligned}
$$

- l is the mean anomaly, n the mean motion & e_t the 'time eccentricity'

- g_{4t}, g_{6t}, f_{4t}, f_{6t}, i_{6t} & h_{6t} are 2PN & 3PN order orbital functions expressible in terms of E, L, m_1 & m_2

- The most accurate & efficient way of solving PN-accurate KE is via adapting Mikkola's method

How to derive GQKP:I

- Obtain 3PN accurate expressions $\dot{r}$ & $r^2\,\dot{\phi}$ in terms of $-2\,E, J, m, \eta$
  $\dot{r} = \mathbf{n} \cdot \frac{\partial \mathcal{H}}{\partial \hat{\mathbf{p}}} \quad r^2\,\dot{\phi} = \left|\mathbf{r} \times \frac{\partial \mathcal{H}}{\partial \hat{\mathbf{p}}}\right|$

- Let $s = 1/r$ & this leads to $\frac{ds}{dt}$ & $\frac{d\phi}{dt}$ being $7^{\rm th}$ degree polynomial in s

- 3PN-accurate $\frac{ds}{dt}$ admits *two positive* real roots, s_+ & s_-
  s_+ & $s_- \rightarrow$ periastron & apastron

- Factorize these roots from $\frac{ds}{dt}$ & $\frac{d\phi}{ds}$ leads to

$$t - t_0 = \int_s^{s_-} \frac{A_0 + A_1\bar{s} + A_2\bar{s}^2 + A_3\bar{s}^3 + A_4\bar{s}^4 + A_5\bar{s}^5}{\sqrt{(s_- - \bar{s})(\bar{s} - s_+)}\,\bar{s}^2} d\bar{s}\,. \tag{45}$$

$$\phi - \phi_0 = \int_s^{s_-} \frac{B_0 + B_1\bar{s} + B_2\bar{s}^2 + B_3\bar{s}^3 + B_4\bar{s}^4 + B_5\bar{s}^5}{\sqrt{(s_- - \bar{s})(\bar{s} - s_+)}} d\bar{s}\,, \tag{46}$$

How to derive GQKP:II

- Expressions for $t - t_0$ & $\phi - \phi_0$ can be integrated using the ansatz $r = a_r(1 - e_r\ \cos u)\ \ \&\tilde{v} = 2\arctan\left[\left(\frac{1+e_r}{1-e_r}\right)^{1/2}\ \tan\frac{u}{2}\right]$

- We also introduce $e_\phi = e_r\left[1 + ...\right]\ \&e_t = e_r\left[1 + ...\right]$ to arrive at 3PN-accurate GQKP

- It follows that

$$a_r = \frac{1}{2}\frac{s_- + s_+}{s_- s_+}\,, \qquad e_r = \frac{s_- - s_+}{s_- + s_+}\,. \tag{47}$$

- The integrals for $t - t_0$ & $\phi - \phi_0$ when evaluated between the limits s_+ & s_- lead to the radial orbital period & the periastron advance rate

  These two are gauge-invariant quantities if expressed in terms of $\mathcal{E}$ & J.

The ? of chaos

- Numerical solution to PN-accurate spinning compact binary dynamics is to be pursued carefully

- It was argued by J. Levin & her co-workers that 2PN accurate spinning compact binary dynamics can be chaotic even when only one of the object spins

- The 2PN-accurate conservative dynamics only contains the leading-order spin-orbit interactions

- Therefore, the dimension of phase-space is $8(3+3+2)$ & hence degrees of freedom is 4

  There are 4 conserved quantities $\mathcal{E}, |\mathcal{L}|, |\mathcal{S}|, \mathcal{L} \cdot \mathcal{S}$

- It should be integrable & hence can not be chaotic !!

GQKP with spins:I

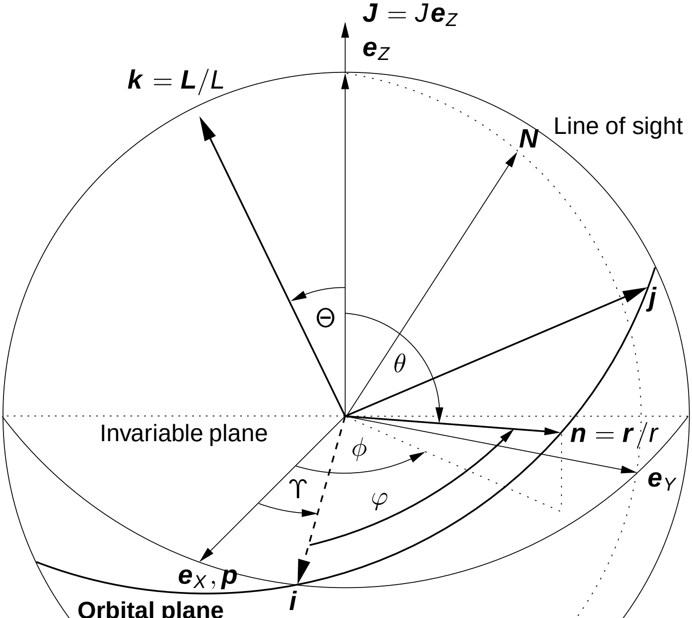
- A parametric solution to PN accurate orbital dynamics that leading order relativistic spin-orbit interactions exists for two specific configurations

  i) $m_1 \neq m_2$, $\boldsymbol{S}_1 \neq 0$ *or* $\boldsymbol{S}_2 \neq 0$ (Single spin case)

  ii) $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are arbitrary but $m_1 = m_2$.

- For these cases, we have Keplerian type parametrization that describes not only the precessional motion of the orbit inside the orbital plane, but also the precessional motions of the orbital plane and the spins themselves

GQKP with spins:II



GQKP with spins:III

- $$\begin{aligned}\boldsymbol{r}(t) &= r(t)\cos\varphi(t)\,\boldsymbol{i}(t) + r(t)\sin\varphi(t)\,\boldsymbol{j}(t)\,,\\ \boldsymbol{L}(t) &= L\boldsymbol{k}(t)\,,\\ \boldsymbol{S}(t) &= J\boldsymbol{e}_Z - L\boldsymbol{k}(t)\,,\end{aligned}$$

  The time evolution of the basic vectors $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ are given by

  $$\begin{aligned}\boldsymbol{i}(t) &= \cos\Upsilon(t)\boldsymbol{e}_X + \sin\Upsilon(t)\boldsymbol{e}_Y\,,\\ \boldsymbol{j}(t) &= -\cos\Theta\sin\Upsilon(t)\boldsymbol{e}_X + \cos\Theta\cos\Upsilon(t)\boldsymbol{e}_Y + \sin\Theta\,\boldsymbol{e}_Z\,,\\ \boldsymbol{k}(t) &= \sin\Theta\sin\Upsilon(t)\boldsymbol{e}_X - \sin\Theta\cos\Upsilon(t)\boldsymbol{e}_Y + \cos\Theta\,\boldsymbol{e}_Z\end{aligned}$$

  Θ, the precessional angle of $\boldsymbol{L}$, $= \frac{S\sin\alpha}{J}$

  $$J = (L^2 + S^2 + 2LS\cos\alpha)^{1/2}$$

  α is the angle between $\boldsymbol{L}$ & $\boldsymbol{S}$.

GQKP with spins:IV

- Time evolution for r, φ & Υ are given by

$$
\begin{aligned}
r &= a_r \left(1 - e_r \cos u\right) , \\
l \equiv n\,(t - t_0) &= u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(v - u) \\
&\quad + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\sin v + \frac{i_{6t}}{c^6}\sin 2v + \frac{h_{6t}}{c^6}\sin 3v , \\
\varphi - \varphi_0 &= (1 + k)v + \left(\frac{f_{4\varphi}}{c^4} + \frac{f_{6\varphi}}{c^6}\right)\sin 2v \\
&\quad + \left(\frac{g_{4\varphi}}{c^4} + \frac{g_{6\varphi}}{c^6}\right)\sin 3v + \frac{i_{6\varphi}}{c^6}\sin 4v + \frac{h_{6\varphi}}{c^6}\sin 5v , \\
\Upsilon - \Upsilon_0 &= \frac{\chi_{\rm so} J}{c^2 L^3}(v + e \sin v)
\end{aligned}
$$

The true anomaly $v = 2 \arctan\left[\left(\frac{1+e_\varphi}{1-e_\varphi}\right)^{1/2} \tan \frac{u}{2}\right]$.

The orbital elements & functions are expressible in terms of E, L, S, m_1, m_2 & α.

Solving PN-accurate dynamics: Summary

- It is worthwhile to pursue a semi-analytic approach to solve PN-accurate orbital dynamics to substantiate purely Numerical results

- It also allows one to specify orbital elements to PN-accurate order & to make sure that numerical results are physically justifiable

Effective one-body approach

Ref. T. Damour; http://arxiv.org/abs/0802.4047

EOB approach:I

Real two-body problem
(two masses m_1, m_2 orbiting around each other)

$$\updownarrow$$

Effective one-body problem
(one test particle of mass m_0 moving in some background metric $g_{\alpha\beta}^{\text{effective}}$)

- The mapping is performed with th help of Hamilton-Jacobi formalism

$$\widehat{S} = -\hat{E}\,\widehat{t} + j\,\phi + \int dr \sqrt{R(r;\hat{E},j)}, \tag{48}$$

Demand that $\widehat{S}$, j & $\mathcal{I}_r$ coincide for the 'real' & 'effective' descriptions

- $\mathcal{E}_{\text{effective}} = f(\mathcal{E}_{\text{real}})$; f is determined in the process of matching.

EOB approach:II

- For a test-particle m_2 in BH of mass m_1

$$\mathcal{E}_t = m_1 - \frac{\boldsymbol{p_1} \cdot \boldsymbol{p_2}}{m_1} \,, \quad \frac{\mathcal{E}_{\rm eff}}{\mu} = \frac{(\mathcal{E}_t)^2 - m_1^2 - m_2^2}{2 m_1 m_2} \,. \tag{49}$$

- Solving for $\mathcal{E}_t$ leads to

$$\boxed{\mathcal{E}_{\rm t} = m \sqrt{1 + 2 \frac{\mu}{m} \frac{\mathcal{E}_{\rm eff} - \mu}{\mu}}.} \tag{50}$$

- This manner one develops $\mathcal{H}^{\rm imp}(r, p_r, j)$

$$\frac{d\hat{r}}{d\hat{t}} = \frac{\partial \hat{\mathcal{H}}^{\rm imp}}{partial \hat{p}_r} \,, \frac{d\phi}{d\hat{t}} = \frac{\partial \hat{\mathcal{H}}^{\rm imp}}{\partial \hat{j}} \equiv \hat{\omega} \tag{51a}$$

$$\frac{d\hat{p}_r}{d\hat{t}} = -\frac{\partial \hat{\mathcal{H}}^{\rm imp}}{\partial \hat{r}} \,, \quad \frac{d\hat{j}}{d\hat{t}} = \mathcal{F}_j \tag{51b}$$

$\mathcal{F}_j$ is the far-zone angular momentum flux & one usually employs Padé approximant version of PN-accurate angular momentum flux for circular inspiral

Hierarchical triplets

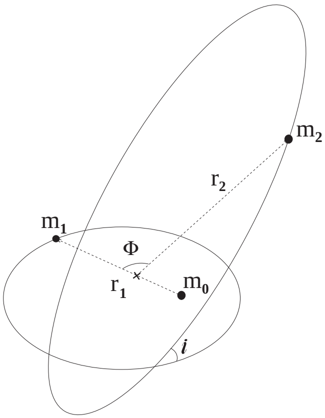
Ref. Ford, Kozinsky & Rasio; 2000

HT in PN approach ?:I

- Discussions about ‘Hierarchical triplets’ appear in dense stellar cluster simulations
- ‘Hierarchical triplets’, consisting of 3 objects (0,1,2), usually modeled to consist of an inner (0,1) & an outer binary (2,3) [‘3‘ $\longrightarrow$ stands for the COM of (1,2) binary]

- If the mutual inclination angle between the two orbital planes is large enough, inner binary may experience oscillations in its eccentricity (Kozai resonances)

  It is due to the time averaged tidal force on the inner binary



HT in PN approach ?:II

- Periastron advance, appearing at 1PN order, should destroy the Kozai resonances..

  The inclusion of the above effect is done in an ad-hoc manner

- Therefore, evolution of ‘Hierarchical triplets’ using fully 1PN-accurate 3-Body Hamiltonian should be of some interest !

- Is it worth pursuing it ?

HT in PN approach ?:III

- For HT, $\mathcal{H} \propto$ sum of two terms representing the two decoupled motions and an infinite series representing the coupling of the orbits.

$$\mathcal{H} \propto \frac{m_0 m_1}{2a_1} + \frac{(m_0 + m_1)m_2}{2a_2} + \frac{1}{a_2} \sum_{j=2}^{\infty} (a_1/a_2)^j \left(\frac{r_1}{a_1}\right)^j \left(\frac{a_2}{r_2}\right)^{j+1} P_j(\cos\Phi), \tag{52}$$

- It is customary to introduce a set of canonical variables $(l_1, l_2), (g_1, g_2), (h_1, h_2)$ and their conjugate momenta $(L_1, L_2), (G_1, G_2), (H_1, H_2)$
  $L_i \propto a_i$

- With the help of a specific canonical transformation, it is possible to introduce new canonical coordinates & momenta such the the Hamiltonian is independent of (l_1, l_2) & hence a_i are conserved..

HT in PN approach ?:IV

- The resulting Hamiltonian is available to $(a_1/a_2)^3$ order & depends only on (e_1, e_2, g_1, g_2) and i, the mutual inclination angle

- It is straightforward, but tedious, to obtain dynamical equations for de_i/dt and dg_i/dt

- To probe the effect of GR, it is customary to add 1PN corrections to dg_1/dt and 2.5PN (Newtonian) radiation reaction contributions to de_1/dt

  We need to assume that the inner binary is isolated

- It should be possible to derive de_i/dt and dg_i/dt at least 1PN order & at quadrupolar order [corrections are $\mathcal{O}((a_1/a_2)^2)$]

- It is doable with 1PN-accurate 3-Body Hamiltonian derived by G. Schäfer in 1987 ! & spin effects can be, in principle, added !!

Conclusions

- PN-accurate dynamics involving compact objects should be useful to the practitioners of N-Body astrophysics, especially while trying to model realistically scenarios leading to potential GW sources