Random numbers

Kankaala (1993): *Monte Carlo Simulations*, CSC Research Reports R03/93, CSC.

Knuth (1981): *Seminumerical Algorithms*, vol 2 of *The Art of Computer Programming*, Addison-Wesley.

Press, Teukolsky, Vetterling, Flannery: *Numerical Recipes*, Cambridge University Press.

"We guarantee that each number is random individually, but we don't guarantee that more than one of them is random."

The sequence of random number (actually pseudorandom numbers) is deterministic. Starting with the same initial value we get the same sequence. Usually the initial value can be chosen. (When testing a program, it may be usefule to use a fixed initial value to be able to repeat the calculations. In the actual work a varying initial value, like the time, is better.)

Linear congruence method

$$X_{i+1} = aX_i + b(\text{mod } m), a, b, m \in \mathbf{N}$$

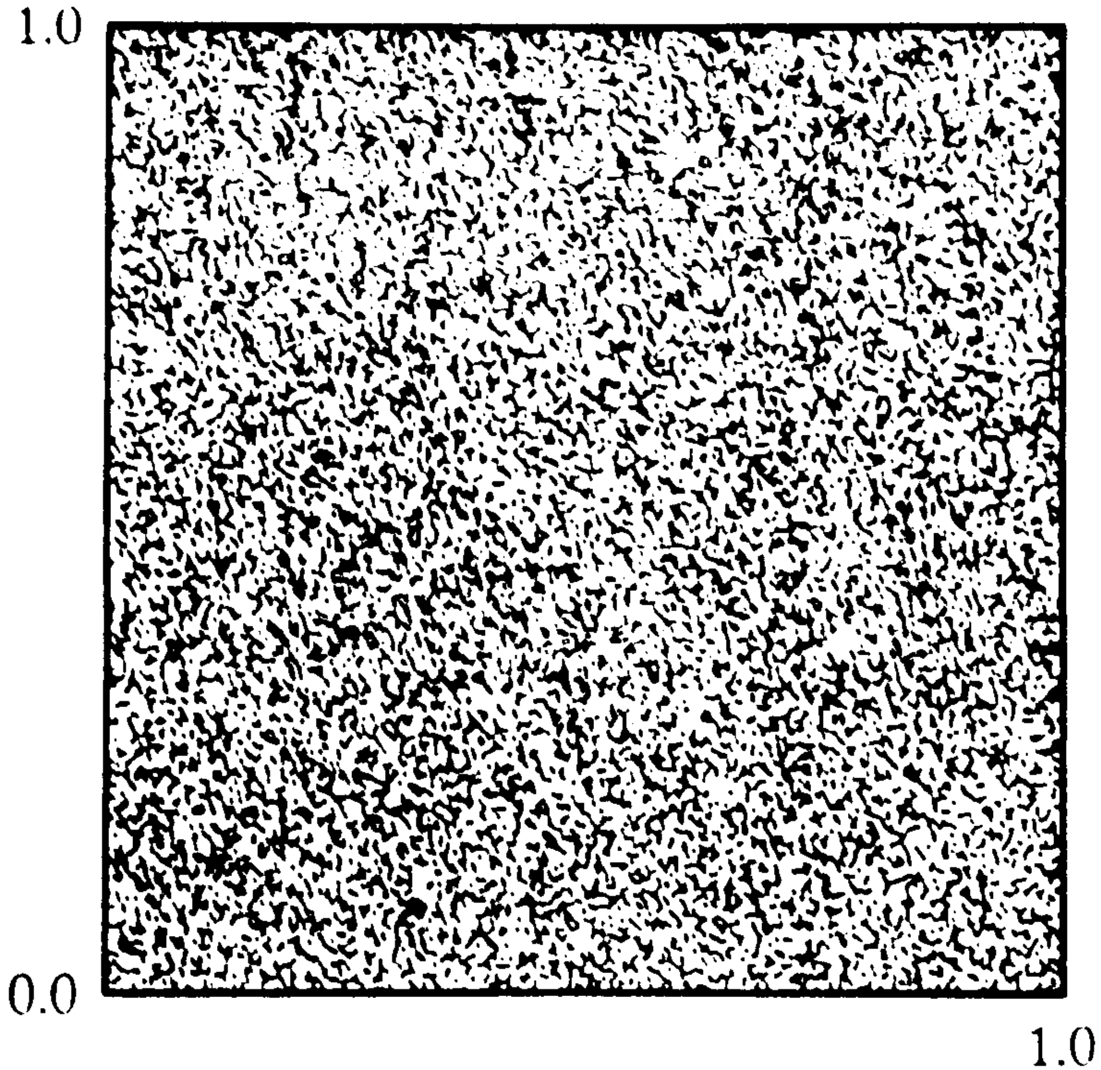
In multiplicative methods $b = 0$, in mixed methods $b > 0$.

The same values will repeat after m numbers at last. If the constants are badly chosen, the sequence can be much shorter.

Correlation of successive numbers: If we take n numbers at a time to give a point in an n-dimensional space, the points do not fill the space uniformly; they are located on $n-1$-dimensional hyperplanes, the number of which is at most $m^{1/k}$, often much less.

The least significant bits of random numbers are least random.

Random numbers must not be split into parts; the parts are not equally random.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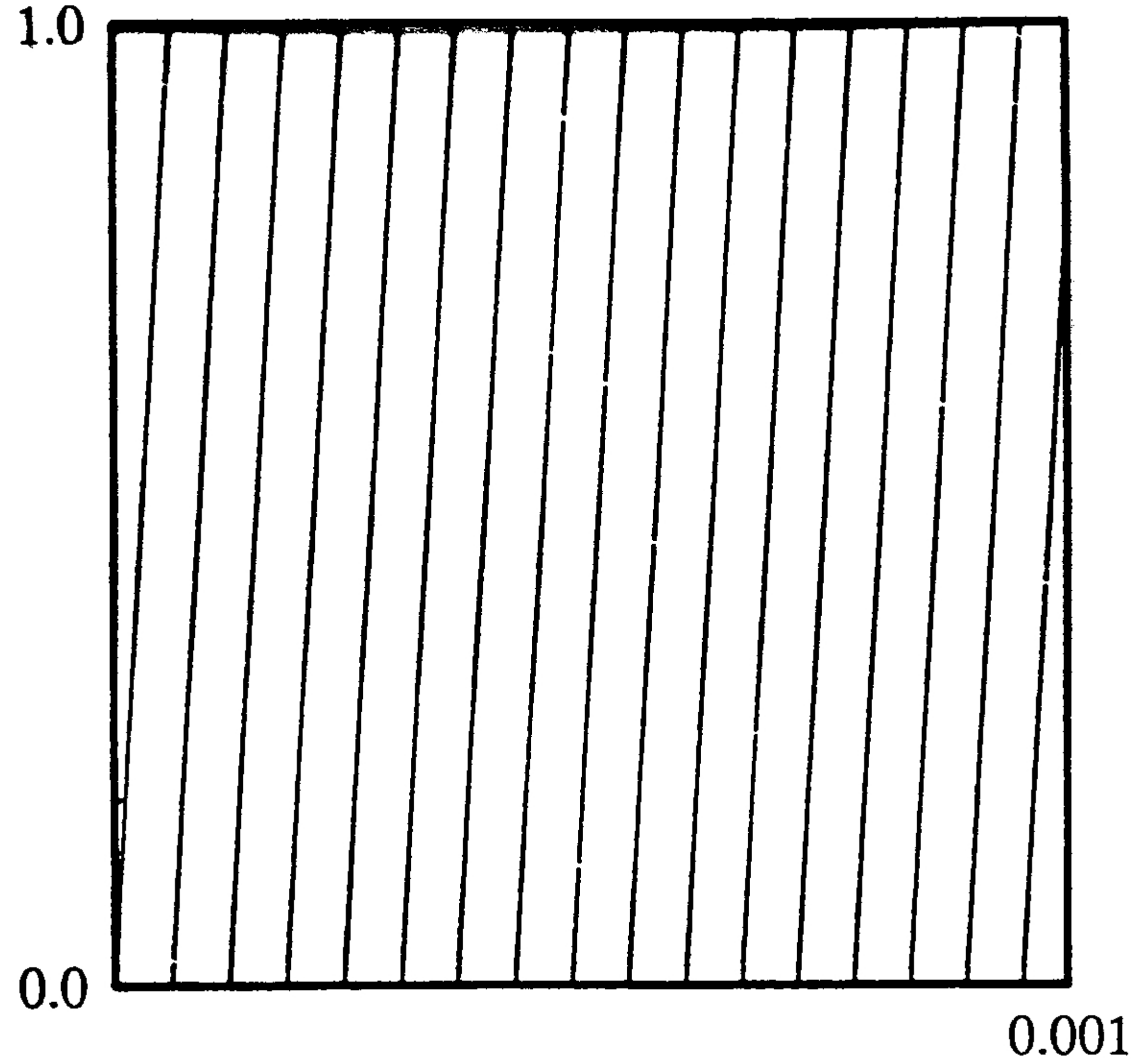


Figure 1: Spatial distribution of 20 000 random number pairs on a unit square. Hyperplane structure is evident for the generator GGL when the horizontal scale is expanded. It should be noted that GGL passed *all* statistical, bit level and physical tests. The failure in the spectral test is a result of the hyperplane structure presented here.

in Fortran 90 random numbers can be generated using the subroutine

```
call random_number (v)
```

Here `v` is an array that will be filled with random numbers. The numbers are uniformly distributed in the range $[0, 1)$.

The rundom number generator can be initialized

```
call random_seed (n)
```

where `n` is an integer.

To study the quality of random numbers, generate a sequence of integers

$$I_i = \lfloor K X_i \rfloor, i = 1, \dots, n$$

- The numbers must be uniformly distributed: the probability of each number should be $1/K$ and it should appear about n/K times in the sequence (can be tested with the χ^2-test)
- Each pair of successive numbers (a, b) must have the same probability K^{-2} etc.
- The numbers X_i and X_{i+k} are not correlated for any $k > 1$.

Other distributions

If X is uniformly distributed in the interval $[0, 1]$, the random variable $a + X(b - a)$ is uniformly distributed in the interval $[a, b]$.

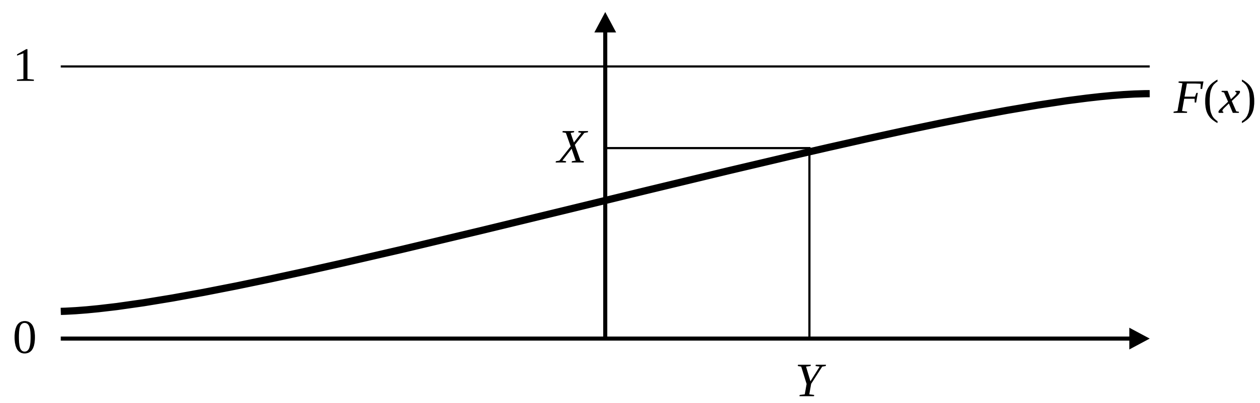
Let F be the cumulative probability distribution of an arbitrary distribution. If X is uniformly dsitributed in $[0, 1]$ and Y is solved from the equation

$$F(Y) = X,$$

we get a variable Y that obeys the given distribution:

$$Y = F^{-1}(X).$$

If the inverse of the cumulative distribution is easily calculated, it can be used to generate random numbers with the given dsitribution.



Exponential distribution

The cumulative distribution of the exponential distribution is

$$F(x) = \int_0^x f(t)\,dt = 1 - e^{-kx},$$

and thus

$$F^{-1}(y) = -\frac{1}{k}\ln(1-y),$$

where k is the expectation of the distribution.

If X is uniformly distributed in $[0, 1]$, so is $1 - X$, and thus

$$Y = -\frac{1}{k}\ln X$$

is exponentially distributed.

- Radioactive decay. If the half-life is T, the probability for a given atom to decay within time t is

$$1 - 2^{-t/T} = 1 - e^{-t\ln 2/T}$$

or the atom will decay in time $[t, t + dt]$ with the probability

$$p(t)\,dt = \frac{\ln 2}{T} e^{-t\ln 2/T}\,dt$$

- Random walk. If the mean free path is λ, the probablity to reach a distance s is

$$p(s)\,ds = \frac{1}{\lambda} e^{-s/\lambda}\,ds$$

Normal distribution

The density function of the (0,1)-normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

There are several ways of generating normally distributed random numbers.

1) Solve numerically the inverse of the cumulative distribution. Too much work, too slow.

2) Sum of random numbers approaches normal distribution. By adding a few uniformly distributed random numbers we get numbers that are nearly normally distributed.

3) Box-Muller method

If x_1 and x_2 are random variables and $y_1 = y_1(x_1, x_2)$ $y_2 = y_2(x_1, x_2)$, the distribution of the variables (y_1, y_2) is

$$p(y_1, y_2)\, dy_1\, dy_2 = p(x_1, x_2) \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| dy_1\, dy_2.$$

Let x_1 be x_2 uniformly distributed in $[0, 1]$ and

$$\begin{aligned} y_1 &= \sqrt{-2\ln x_1}\cos 2\pi x_2, \\ y_2 &= \sqrt{-2\ln x_1}\sin 2\pi x_2, \end{aligned}$$

or

$$\begin{aligned} x_1 &= e^{-(y_1^2+y_2^2)/2}, \\ x_2 &= \frac{1}{2\pi}\arctan\frac{y_2}{y_1}. \end{aligned}$$

The Jacobian determinant is

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{\partial(x_1)}{\partial(y_1)} & \frac{\partial(x_1)}{\partial(y_2)} \\ \frac{\partial(x_2)}{\partial(y_1)} & \frac{\partial(x_2)}{\partial(y_2)} \end{vmatrix} =$$

$$-\left(\frac{1}{\sqrt{2\pi}}e^{-y_1^2/2}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-y_2^2/2}\right).$$

This is the product of two density functions of normal distributions: Both y_1 and y_2 are normally distributed.

Rejection method

Let p be the density function of the desired distribution and f a reference function such that $f(x) \geq p(x)$ for all x.

Generate a random number x obeying the distribution f $(x = F^{-1}(X))$.

Take another random number y, uniformly distributed in $[0, f(x)]$. If $y < p(x)$ the number is accepted, otherwise the number is rejected and the process repeated.

Random walk

For example, radiative transfer in a medium.

The mean free path is known. The distance a photon will travel before hitting a particle is exponentially distributed.

The direction of the scattered photon is given by the scattering function.

Even a small number of photons will give a rough distribution of the outgoing radiation. The distribution can be made more accurate as needed by tracing more photons.

Ray tracing, ray casting

Usually the geometry is known. The point where the ligh beam will hit a surface can be solved in a deterministic way. The scattered light can have different components:

- specular (mirror like) reflection

- diffuse part (according to some scattering law)

- the object itself may radiate (luminence)

Each beam can be traced independently from the others. Methods well adapted to parallel computers.